# Rational Orthonormal Bases

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## Overview

Application: Pedagogical. Teaching QR factorization.
Problem: Most textbooks use examples containing square-root calculations.
Aim: Create examples requiring *only* rational arithmetic **at each stage**.
Connection to Pythagorean $n$-tuples: The columns of a rational Q matrix are Pythagorean.

## Pythagorean N-tuple and Vector

A Pythagorean **n-tuple** is a set of n-positive integers $x_i$ such that

$$\sum_{i=1}^{n-1}(x_i)^2 = (x_n)^2.$$

A **vector** $x = [x_1, ..., x_m]$ is Pythagorean if $\forall k$, $x_k \in \mathbb{Z}$ and $\|x\|_2 \in \mathbb{N}$.

## Orthogonalization

The significance of Pythagorean vectors can be observed by taking $a, b, c$ as a Pythagorean triple, then

$$Q = \begin{bmatrix} a/c & -b/c \\ b/c & a/c \end{bmatrix}$$

is **orthonormal**.

For practicality, $A$ and $Q$ matrices are written in terms of their columns, but the $R$ matrix with its elements.

$$\begin{aligned} A &= [a_1\ a_2\ ...\ a_n] \\ &= QR \\ &= [q_1\ q_2\ ...\ q_n] \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ & r_{22} & \dots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix} \end{aligned} \qquad (1)$$

Then,

$$a_k = \sum_{i=1}^{k} q_i r_{ik} \qquad (2)$$

where the scalar $r_{ik}$ is placed after the vector $q_i$ to line up the indices.

## Gram-Schmidt

The classical Gram-Schmidt process in matrix notation starts from $k = 1$ to $n$

$$b_k = a_k - \sum_{j=1}^{k-1}(a_k.q_j)q_j, \qquad (3)$$

$$q_k = \frac{b_k}{||b_k||}, \qquad (4)$$

where, the norm is the 2-norm.

### *Theorem*

*If $A$ obeys (1) and (2) and the $q_k$ and $r_{ik}$ are rational then all quantities in (3) and (4) are rational.*

***Proof.*** By induction, the values obtained by the reduction are denoted as $\hat{q}_i$ and $\hat{r}_{ik}$. For $k = 1$, $b_1 = a_1 = q_1 r_{11}$ and hence $\hat{q}_1 = q_1 \operatorname{sgn}(r_{11})$ and $\hat{r}_{11} = |r_{11}|$. For general k,

$$b_k = a_k - \sum_{j=1}^{k-1}(a_k.\hat{q}_j)\hat{q}_j = a_k - \sum_{j=1}^{k-1}(a_k.q_j)q_j$$

$$= \sum_{i=1}^{k} q_i r_{ik} - \sum_{j=1}^{k-1}\sum_{i=1}^{k}(q_i r_{ik}).q_j q_j = q_k r_{kk}$$

hence $\hat{q}_k = q = k \operatorname{sgn}(r_{kk})$ and $\hat{r}_{kk} = |r_{kk}|$ and all **inner products** are rational.

## Householder Transformations

If $v$ is a non-zero vector in $R^n$, the householder transformation of $v$ is its reflection with respect to a hyperplane $v$ in $R^n$ orthogonal to $v$, through the origin represented by the outer product of $v$ with itself $vv^T$, then the $n \times n$ orthogonal matrix

$$H_{v^\perp} = I - \frac{2vv^T}{v^T v} \qquad (5)$$

is called the **Householder matrix**.

### *Theorem*

*Let $v_1 = [x_1, \dots, x_n]$ and $v_2 = [y_1, \dots, y_n]$ be Pythagorean. Let $v_1$ be orthogonal to $v_2$. Then the $(n-1)$-dimensional vector*

$$w = (X - x_1)[y2, \dots, y_n] + y_1[x_2, \dots, x_n],$$

*where $X = \|v_1\|$ and $Y = \|v_2\|$ is Pythagorean and $\|w\| = (\|v_1\| - x_1)\|v_2\|$*

## Proof

$$\begin{aligned} \|w\|^2 &= \sum_{k=2}^{n}\left[(X - x_1)y_k + y_1 x_k\right]^2 , \\ &= (X - x_1)^2 \sum_{k=2}^{n} y_k^2 + 2(X - x_1)y_1 \sum_{k=2}^{n} x_k y_k \\ &\quad + y_1^2 \sum_{k=2}^{n} x_k^2 , \\ &= (X - x_1)^2(Y^2 - y_1^2) + 2(X - x_1)y_1(-x_1 y_1) \\ &\quad + y_1^2(X^2 - x_1^2) , \\ &= (X - x_1)[XY^2 - Xy_1^2 - x_1 Y^2 + x_1 y_1^2 \\ &\quad - 2x_1 y_1^2 + y_1^2 X + x_1 y_1^2] , \\ &= (X - x_1)\left[XY^2 - Xy_1^2 - x_1 Y^2 + y_1^2 X\right] \\ &= (X - x_1)[XY^2 - x_1 Y^2] . \end{aligned}$$

### Example

Construct a matrix using a known $Q$ matrix (see below for source) and a user-chosen $R$ matrix.

$$A = \begin{bmatrix} \frac{3}{13} & \frac{4}{5} & \frac{36}{65} \\ \frac{4}{13} & -\frac{3}{5} & \frac{48}{65} \\ \frac{12}{13} & 0 & -\frac{5}{13} \end{bmatrix} \begin{bmatrix} 13 & 26 & 13 \\ 0 & 5 & -5 \\ 0 & 0 & 65 \end{bmatrix} = \begin{bmatrix} 3 & 10 & 35 \\ 4 & 5 & 55 \\ 12 & 24 & -13 \end{bmatrix}$$

The Householder matrix $H_1$ is then

$$H_1 = \begin{bmatrix} \frac{3}{13} & \frac{4}{13} & \frac{12}{13} \\ \frac{4}{13} & \frac{57}{65} & -\frac{24}{65} \\ \frac{12}{13} & -\frac{24}{65} & -\frac{7}{65} \end{bmatrix} .$$

Now, computing the product of $H_1 A$ gives something intriguing.

$$A_1 = H_1 A = \begin{bmatrix} 13 & 26 & 13 \\ 0 & -\frac{7}{5} & \frac{319}{5} \\ 0 & \frac{24}{5} & \frac{67}{5} \end{bmatrix}$$

Householder continues, using the submatrix

$$\begin{bmatrix} -\frac{7}{5} & \frac{319}{5} \\ \frac{24}{5} & \frac{67}{5} \end{bmatrix}$$

## Example cont.

We observe that the first column now contains the Pythagorean vector $[7, 24]$. Therefore the Householder transformation remains rational. This is because the theorems above and below apply.

### *Theorem*

*Given an integer vector $[p_1, ..., p_n]$ not necessarily Pythagorean, then*

$$V_n = [2p_1^2 - \sum_{i=1}^{n} p_i^2, 2p_1p_2, ..., 2p_1p_n]$$

*is a Pythagorean vector.*

## Interesting Q Matrices

For this work to be useful to instructors of Linear Algebra, they must have a source of rational $Q$ matrices. We are planning to offer such a collection. When making a collection, one cannot resist remarking on patterns in the matrices collected. Behold:

*Circulant-like Matrices*

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} , \quad \frac{1}{7}\begin{bmatrix} 1 & -4 & -4 & -4 \\ 4 & 1 & -4 & 4 \\ 4 & 4 & 1 & -4 \\ 4 & -4 & 4 & 1 \end{bmatrix}$$

*Permutations of a vector*

$$\frac{1}{5}\begin{bmatrix} 1 & -2 & 2 & 4 \\ 2 & 1 & 4 & -2 \\ 2 & 4 & -1 & 2 \\ 4 & -2 & -2 & -1 \end{bmatrix}$$

*Block patterns*

$$\frac{1}{4}\begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & -3 & 1 & 1 & 1 \\ 2 & 1 & -3 & 1 & 1 \\ 2 & 1 & 1 & -3 & 1 \\ 2 & 1 & 1 & 1 & -3 \end{bmatrix}$$

*A Bohemian matrix!*

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$