Non-commutative Riquier Theory in Moving Frames of Differential Operators

Francois Lemaire

Department of Computer Science, The University of Western Ontario, London, Ontario, N6A 5B7, Canada

Greg Reid and Yang Zhang

Department of Applied Mathematics, The University of Western Ontario, London, Ontario, N6A 5B7, Canada

Abstract

Moving frames chosen to be invariant under a known Lie group $\mathcal{G}$ provide a powerful generalization of the idea of choosing $\mathcal{G}$-invariant coordinates to cases where $\mathcal{G}$-invariant coordinates do not exist. Such $\mathcal{G}$-invariant formulations are of great current interest in areas such as Geometric Integration where $\mathcal{G}$-invariant integrators (e.g. symplectic integrators), can often substantially outperform non-invariant integrators. They are also of substantial interest in applications where one would like to factor out a known group.

One form of classical existence and uniqueness theory for analytic PDE referred to (standard) commuting partial derivatives is that of Riquier, which was formulated and generalized by Rust using a Gröbner style development.

We extend the Rust-Riquier existence and uniqueness theory to analytic PDE written in terms of moving frames of non-commuting Partial Differential Operators (PDO). The main idea for the theoretical development is to use the commutation relations between the PDO to place them in a standard order. This normalization is exploited to generalize the corresponding steps of the commuting Rust-Riquier Theory to the non-commuting case.

Given an equivalence group $\mathcal{G}$ Lisle has given a $\mathcal{G}$-invariant method for determining the structure of Lie symmetry groups of classes of PDE. Lisle's method for such group classification problems was illustrated on a number of challenging examples, which lead to unmanageable expression explosion for computer algebra programs using the standard (commuting) frame. He obtained new results, which for want of an existence and uniqueness theorem for PDE in non-commuting frames, had to be individually checked. We provide an existence and uniqueness theorem making rigorous the output from Lisle's method. For the finite parameter group case, the output is reformulated in terms of the integration of a system of Frobenius type, which can be numerically integrated by integrating an ODE system along a curve.

AMS Classification: 35N10, 58J70, 53A55, 13P10, 12H05.

1 Introduction

We consider analytic systems of partial differential equations (PDE) with independent variables $x = (x_1, x_2, ..., x_m)$ and dependent variables $u = (u^1, ..., u^n)$. We give existence and uniqueness theorems for systems written in terms of differential operators $\widetilde{\partial}_i$ of the form:

$$\widetilde{\partial}_i = \sum_{j=1}^{m} a_{ij}(x,u)\frac{\partial}{\partial x_j}, \quad i = 1, ..., m, \quad \det(a_{ij}(x,u))_{m\times m} \neq 0. \tag{1}$$

An easy computation shows that the $\widetilde{\partial}_i$ satisfy commutation relations of the form:

$$[\,\widetilde{\partial}_i, \widetilde{\partial}_j\,] = \sum_k \gamma_{ij}^k \widetilde{\partial}_k\ , \quad 1 \leq i \leq j \leq m, \tag{2}$$

where the γ_{ij}^k are so-called structure functions of x, u and first order derivatives of u. Thus the $\widetilde{\partial}_i$ are generally non-commutative in comparison to the usual commuting partial derivatives $\frac{\partial}{\partial x_j}$ (which will be abbreviated as ∂_{x_j} or ∂_j).

Any system of PDE can be written in terms of such a system of non-commutative operators by inverting the relation (1).

It is not immediately clear why one would want to give up commutativity to express PDE in terms of non-commuting operators (in a so-called *moving frame of differential operators*). The motivation is that a non-commuting frame may enjoy properties not shared by the standard commuting frame. For example such properties might be geometrical properties such as invariance under a certain Lie group $\mathcal{G}$. A special case is that of using polar coordinates for cylindrically invariant problems (where the operators are $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial r}$ and in fact commute). A classic example of a moving frame which does not yield a global coordinate system is the existence of a global coordinate system on the Torus. This process of choosing appropriate coordinates to avoid unnecessarily complicated expressions, has a long history. Given a $\mathcal{G}$-invariant problem, however, it is not possible to always choose $\mathcal{G}$-invariant coordinates.

Cartan, with his method of moving frames, found a significant and far-reaching generalization of such ideas (see [1, Chapt 5] for historical remarks and also the foundational work of Tresse [33]). More recent works include those of Griffiths

Email addresses: `lemaire@orcca.on.ca` (Francois Lemaire), `reid, yzhang26@uwo.ca` (Greg Reid and Yang Zhang).

[10], and a new more general constructive approach was given to Moving Frame Theory by Fels and Olver [7,8]. Given an arbitrary Lie group $\mathcal{G}$ the power of the general method of moving frames is that under fairly weak conditions, on some sufficiently prolonged space, a $\mathcal{G}$-invariant frame exists.

A major motivation for our work was provided by work of Lisle [16] (also see [17]). That work concerned the computation and exploitation of Lie symmetries of classes of differential equations. For example, in modelling diffusion, one may be interested in classes of nonlinear diffusion equations of the form

$$u_t = \Big(K(u)u_x\Big)_x \,, \tag{3}$$

where the diffusion is assumed to be nonlinear ($K_u(u) = \dot{K}(u) \neq 0$). A common objective is to determine functional forms of the diffusion coefficient $K(u)$, capable of modelling physically important diffusion processes, for which exact solutions of the diffusion PDE can be found. Lie group classification methods can in theory determine the $K(u)$ for which such nonlinear diffusion PDE have large symmetry groups, and give procedures for identifying corresponding classes of exact solutions.

Algorithms [26,27] based on commuting partial derivatives, exist for identifying the size and structure of the symmetry groups of classes of PDE such as (3). Computer implementations of the above algorithms using commuting partial derivatives rely on differential elimination packages such as the *RifSimp*, *Diffalg* and *Diffgrob* packages in Maple. These packages manipulate the defining equations for infinitesimal Lie symmetries of the physical PDE of interest. These defining equations are overdetermined linear homogeneous PDE with coefficients depending on the so-called classification functions (e.g. the $K(u)$ in the PDE above). We direct the reader to the review article of Hereman on symbolic packages for differential equations [12]. These differential generalizations of Gröbner Bases [3], when applied to such systems, typically build up coefficients involving derivatives of the classification functions. These coefficients can become so large and complicated [16], that they can fail to terminate in the available time and memory. This problem persists today, despite considerable progress in both computer speed/memory and improvements in differential elimination algorithms based on commuting PDO.

The idea of Lisle's method [16,17] to address the expression explosion often encountered in such classification problems, was to exploit easily determined equivalence transformations that mapped one member of such a class to another member (paradoxically, an easier problem, than that of determination of symmmetries mapping a member to itself). For example it is easily seen that the class of transformations:

$$x = \beta x', \qquad t = t', \qquad u = \gamma u' + \alpha, \qquad \beta, \gamma \neq 0 \tag{4}$$

map the diffusion equation to $\gamma u'_{t'} = \gamma\beta^{-2}\Big(K(\gamma u' + \alpha)u'_{x'}\Big)_{x'}$. Hence the coefficient $K(u)$ is mapped to a new coefficient, given by $K'(u) = \beta^{-2}K(\gamma u' + \alpha)$.

Lisle's method [16,17] exploits such equivalence transformations *during* the process of finding symmetries by recasting the equations for symmetries in a form which is invariant under the equivalence group. He was able to complete group classification problems, which could not be done by computer algebra methods based on commuting derivations. For these and other non-trivial examples, the reader is directed to [16,17].

Another motivation for our work, is the revitalized interest in Cartan's method of moving frames, its applications and generalizations (see [21] and the review paper [23]). Applications include: various forms of equivalence problem such deciding when two objects are equivalent [4] under the projective group (a fundamental problem in computer vision), and deciding when two differential equations are equivalent by a change of variables. The design of group invariant numerical methods is also an important application which falls under the new area of geometric integration [11].

In his review Olver [23, page 2-3] states that "... any serious application ... will rely on computer algebra", and further that "large scale applications ... will require the development of a suitable noncommutative Gröbner basis theory for such algebras, complicated by the non-commutativity of the invariant differential operators ...".

In this paper we give existence and uniqueness theorems for systems of analytic PDE in a certain form with respect to moving frames of differential operators. This analytic non-commutative Gröbner-style theory is a partial answer to Olver's open problem stated above. It allows nonlinearity which is not present in the linear differential-algebraic theory we presented in [9]. That linear theory did however allow the coefficient rings to be noncommutative which is relevant in non-commutative physical field theories having for example, non-commutative matrix coefficients.

We briefly discuss the dichotomy between such analytic and differential algebraic approaches. Rust [30,31] has given a Gröbner style development of Riquier Theory and generalized this to the nonlinear case. This work has helped bring analytic differential elimination methods (in the spirit of Riquier) and differential-algebraic approaches (as initiated by Ritt and Kolchin) closer together. Still neither theory strictly contains the other. Specializing analytic functions to polynomials, does not yield all the results in differential algebra. Conversely the setting of Differential Algebra at this time, is too narrow to yield the full generality of the analytic approaches. Joint work with Hubert is ongoing to try to bring both approaches into a common theoretical setting. For the moment parallel developments seem necessary.

The main idea of our theory is to use the commutation relations to put the operators in a normal order modulo lower order terms. Then a non-commutative theory is built by mimicking the commutative theory across leading order derivatives (the commutative theory of Rust [31]). In particular we exploit a bijection between the commuting partial derivatives and the non-commutative differential operators $\widetilde{\partial}_i$ (see Lisle and Reid (2000) [17, Appendix A] in which the treatment detailed in the current article is first sketched). We give a rigorous foundation and justification for the group classification-equivalence method of Lisle [16,17]. Our results are not quite as general as those that would be required for a complete treatment of Olver's open problem, but are at the same time applicable to frames of operators enjoying geometric features other than $\mathcal{G}$-invariance. In particular a moving frame in Olver's approach is a $\mathcal{G}$-invariant map from a manifold to a group. The difficulty of establishing a rigorous noncommutative Gröbner basis theory for the moving frames case has become apparent since the seminal work of Mansfield [18], which produces interesting results, but similarly to the less ambitious work of Lisle, lacks an existence and uniqueness theorem.

2 An Example - the Nonlinear Diffusion Equation

As a running example we treat the group classification problem for the nonlinear diffusion equation (3). That problem is to identify for all possible functional forms of the diffusion coefficient and the corresponding Lie symmetry algebras of vector fields $\mathbf{X} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u$ leaving invariant (3).

The components ξ, τ, η of the symmetry vector field obey defining equations which can be automatically produced by many computer algebra packages

$$\tau_x = \tau_u = \xi_u = \eta_{uu} = 0 \tag{5a}$$
$$K(2\xi_x - \tau_t) - \dot{K}\eta = 0 \tag{5b}$$
$$K(2\eta_{xu} - \xi_{xx}) + 2\dot{K}\eta_x + \xi_t = 0 \tag{5c}$$
$$K\eta_{xx} - \eta_t = 0. \tag{5d}$$

For given $K(u)$, this is an overdetermined linear homogeneous system. This system is simple enough to have all of its cases analyzed using differential elimination packages based on commuting PDO and is used for purposes of illustration (see [24] for first time that this PDE was group classified).

We seek to write the defining equations (5) for its symmetries in a form invariant under the action of the equivalence group (4).

Following the method of Lisle [16] leads to the following frame:

$$\widetilde{\partial}_1 := K^{1/2}\partial_x \ , \ \ \widetilde{\partial}_2 := \partial_t \ , \ \ \widetilde{\partial}_3 := K/\dot{K}\,\partial_u \ . \tag{6}$$

The reader can verify that this frame of PDO is invariant under the equivalence group For example $\widetilde{\partial}_1 = K^{1/2}\partial_x = \beta(K')^{1/2}\frac{1}{\beta}\partial_{x'} = (K')^{1/2}\partial_{x'}$. Lisle's method also requires introducing new infinitesimals defined by $\theta^1\widetilde{\partial}_1 + \theta^2\widetilde{\partial}_2 + \theta^2\widetilde{\partial}_2 = \xi\partial_x + \tau\partial_t + \eta\partial_u$ yielding

$$\theta^1 := K^{-1/2}\xi \ , \ \ \theta^2 := \tau \ , \ \ \theta^3 := \dot{K}/K\,\eta \tag{7}$$

which the reader can verify are invariant. Lisle's method also yields the scalar equivalence group invariant

$$J := \frac{K\ddot{K}}{\dot{K}^2} - 1 \ , \ \ \widetilde{\partial}_1 J = 0 \ , \ \ \widetilde{\partial}_2 J = 0 \tag{8}$$

Computation of the structure relations for the frame by making the replacements (6) yields

$$[\widetilde{\partial}_1, \widetilde{\partial}_3] = -\tfrac{1}{2}\widetilde{\partial}_1 \ , \ \ [\widetilde{\partial}_1, \widetilde{\partial}_2] = 0 \ , \ \ [\widetilde{\partial}_2, \widetilde{\partial}_3] = 0 \ . \tag{9}$$

The defining system (5) becomes

$$\begin{array}{lll} \widetilde{\partial}_3\theta^1 + \frac{1}{2}\theta^1 = 0 & \widetilde{\partial}_1\theta^2 = 0 & \widetilde{\partial}_1\widetilde{\partial}_1\theta^3 - \widetilde{\partial}_2\theta^3 = 0 \\ & \widetilde{\partial}_2\theta^2 - 2\widetilde{\partial}_1\theta^1 + \theta^3 = 0 & \widetilde{\partial}_1\widetilde{\partial}_3\theta^3 - \frac{1}{2}\widetilde{\partial}_1\widetilde{\partial}_1\theta^1 - (J-1)\widetilde{\partial}_1\theta^3 + \frac{1}{2}\widetilde{\partial}_2\theta^1 = 0 \\ & \widetilde{\partial}_3\theta^2 = 0 & \widetilde{\partial}_3\widetilde{\partial}_3\theta^3 - J\widetilde{\partial}_3\theta^3 - \widetilde{\partial}_3 J\theta^3 = 0 \end{array} \tag{10}$$

For example $\xi_u = 0$ implies that $\left(\frac{\dot{K}}{K}\widetilde{\partial}_3\right)\left(K^{1/2}\theta^1\right) = 0$ and that $\widetilde{\partial}_3\theta^1 + \frac{1}{2}\theta^1 = 0$ by using $\widetilde{\partial}_3 K = K$.

Our aim with the above system was not to give a detailed explanation of how Lisle's method (which is described elsewhere [16,17]), but instead to give the reader some insight, on the origin of such systems written in terms of non-commuting PDO.

The goal of the rest of the paper is to develop an existence and uniqueness theory for analytic systems such as (10) which are expressed in terms of non-commuting PDO.

3 Derivations

Let $\mathbb{F}$ be a field ($\mathbb{R}$ or $\mathbb{C}$ in practice) with characteric zero, $x = (x_1, \cdots, x_m)$ be the independant variables and $u = (u^1, \cdots, u^n)$ be the dependant variables for a system of PDE.

In the usual commutative approaches to differential algebra and differential elimination theory [30,2], a set of indeterminates corresponding to the partial derivatives is defined:

$$\Omega = \{v_\alpha^i \mid \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m, i = 1, \cdots, n\} .$$

Each indeterminate of Ω corresponds to a partial derivative by:

$$v_\alpha^i \leftrightarrow (\partial_m)^{\alpha_m} \cdots (\partial_1)^{\alpha_1} u^i(x_1, \cdots, x_n) := \partial^\alpha u^i(x_1, \cdots, x_n) .$$

As usual the commutative total derivative operators are then introduced to act on members of Ω by a unit increment of the i-th index of their vector subscript:

$$D_i v_\alpha^k := v_\beta^k .$$

where $\beta = (\alpha_1, ..., \alpha_i + 1, ..., \alpha_n)$. The usual (commutative) total derivative $D_{x_i} \equiv D_i$ action on functions of $\{x\} \cup \Omega$ is then given by:

$$D_i = \partial_i + \sum_{v \in \Omega} (D_i v) \frac{\partial}{\partial v} . \tag{11}$$

The corresponding construction for the non-commutative case is as follows.

We suppose that there are n derivations $\tilde{\partial}_1, ..., \tilde{\partial}_n$ which act on formal power series in the x_i with coefficients in $\mathbb{F}$. The derivation operators do not necessarily commute, that is, $\tilde{\partial}_i \tilde{\partial}_j \neq \tilde{\partial}_j \tilde{\partial}_i$ (e.g. see (9)).

Theorem 3.1 *Since the $\tilde{\partial}_i$ are derivations, they are of the form:*

$$\tilde{\partial}_i = \sum_{j=1}^{m} a_{ij} \partial_j \text{ where } a_{ij} = \tilde{\partial}_i(x_j). \tag{12}$$

PROOF. A derivation $\tilde{\partial}_i$ satifies $\tilde{\partial}_i(f + g) = \tilde{\partial}_i(f) + \tilde{\partial}_i(g)$ and $\tilde{\partial}_i(f\, g) = \tilde{\partial}_i(f)\, g + f\, \tilde{\partial}_i(g)$ where f and g are any power series. Using those two properties, a derivation on the formal power series is uniquely defined by its action on the variables x_k. Since $\sum_{j=1}^m a_{ij} \partial_j$ is a derivation (a linear combination of derivations is a derivation) and satifies $\sum_{j=1}^m a_{ij} \partial_j(x_k) = a_{ik} = \tilde{\partial}_i(x_k)$, equation (12) follows. □

Consider the set of indeterminates

$$\tilde{\Omega} = \{\tilde{v}_\alpha^i \mid \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m, i = 1, \cdots, n\} .$$

Each indeterminate of this set corresponds to a derivation by:

$$\tilde{v}_\alpha^i \leftrightarrow (\tilde{\partial}_m)^{\alpha_m} \cdots (\tilde{\partial}_1)^{\alpha_1} u^i(x_1, \cdots, x_n) := \tilde{\partial}^\alpha u^i(x_1, \cdots, x_n) .$$

In contrast to the commutative case this correspondence only gives a subset of the set of all derivations. However the commutation relations will enable us to extend this correspondence to the whole set.

Note that the full set of derivations of dependant variables of order r contains nm^r members which is far greater than the corresponding number of r-order derivations of the form above (which is $n\binom{r+m-1}{r}$).

To be able to apply a reduction process (described in section 5), and prove uniqueness and existence in Theorem 7.4, we impose:

Blanket Hypothesis 3.2 (Analyticity-Invertibility Assumption) *Throughout this paper we assume that the matrix $(a_{ij}(x,u))$ is an analytic function of x, u with coefficients in $\mathbb{F}$ and is invertible in the domain we are interested in.*

With Hypothesis 3.2, we have the following commutation rules:

$$\widetilde{\partial}_i\widetilde{\partial}_j - \widetilde{\partial}_j\widetilde{\partial}_i = \sum_{k=1}^{m} b_{ij}^k \widetilde{\partial}_k \tag{13}$$

where the b_{ij}^k are analytic functions of x, u and first derivatives of u with values in $\mathbb{F}$.

PROOF. By replacing in $\widetilde{\partial}_i\widetilde{\partial}_j - \widetilde{\partial}_j\widetilde{\partial}_i$ the expressions $\widetilde{\partial}_i$ and $\widetilde{\partial}_j$ given by (12), we get a linear combination of the ∂_i (order 2 derivations are cancelled). By inverting the matrix $(a_{ij}(x,u))$, each ∂_k is itself a linear combination of the $\widetilde{\partial}_k$'s. Thus $\widetilde{\partial}_i\widetilde{\partial}_j - \widetilde{\partial}_j\widetilde{\partial}_i$ is a linear combination of $\widetilde{\partial}_k$'s. □

Nontrivial examples of moving frames of PDO can be found in Lisle and Reid [17], Mansfield [18] and Spivak [32].

From (12) it is natural to define the (non-commuting) formal total derivation by:

$$\widetilde{D}_i = \sum_{j=1}^{m} a_{ij}(x,u) D_j \tag{14}$$

By the commutation rule (13), any $\widetilde{D}_j\tilde{v}$ can be rewritten (normalized) as a function of $\{x\} \cup \widetilde{\Omega}$. Assuming this normalization gives as a consequence of (11),(14)

$$\widetilde{D}_i = \widetilde{\partial}_i + \sum_{\tilde{v}\in\widetilde{\Omega}} (\widetilde{D}_i\tilde{v}) \frac{\partial}{\partial \tilde{v}} \tag{15}$$

on functions of $\{x\}\cup\widetilde{\Omega}$. As a consequence we can now extend our normalization process to functions of $\{x\} \cup \widetilde{\Omega}$.

Blanket Hypothesis 3.3 (Normalization Assumption for Derivations) *In this article, each time a derivation is applied to to a function of* $\{x\} \cup \widetilde{\Omega}$ *we assume that the commutation rules are applied to get an expression only involving elements of* $\widetilde{\Omega}$.

For example, in system (10) all of the derivations are in $\widetilde{\Omega}$ except for $\widetilde{\partial}_1\widetilde{\partial}_3\theta^3$. So using the commutation relation $[\widetilde{\partial}_1, \widetilde{\partial}_3] = -\frac{1}{2}\widetilde{\partial}_1$ in (9) we can replace $\widetilde{\partial}_1\widetilde{\partial}_3\theta^3$ in (10) by $\widetilde{\partial}_3\widetilde{\partial}_1\theta^3 - \frac{1}{2}\widetilde{\partial}_1\theta^3$ so that the fifth equation of (10) is replaced with

$$\widetilde{\partial}_3\widetilde{\partial}_1\theta^3 - \tfrac{1}{2}\widetilde{\partial}_1\widetilde{\partial}_1\theta^1 - (J - \tfrac{1}{2})\widetilde{\partial}_1\theta^3 + \tfrac{1}{2}\widetilde{\partial}_2\theta^1 = 0\ . \tag{16}$$

Denoting $\widetilde{\partial}^\alpha = \widetilde{\partial}_m^{\alpha_m} \cdots \widetilde{\partial}_1^{\alpha_1}$ and $\partial^\alpha = \partial_m^{\alpha_m} \cdots \partial_1^{\alpha_1}$ where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, we have the following property:

Theorem 3.4 (Bijection between derivations and partial derivatives) *Each derivation operator can be expressed as an invertible linear function of partial differential operators.*

PROOF. Using relation (12), any derivation monomial $\widetilde{\partial}^\alpha$ can be rewritten as a linear combination of ∂^α. Conversely, any derivation monomial ∂^α can be rewritten as a linear combination of $\widetilde{\partial}^\alpha$'s using Hypotheses 3.2 and 3.3. □

4 Rankings

As with any Gröbner style theory, rankings play a fundamental role.

Suppose $\prec$ is a total order on the set of (normalized) derivations $\widetilde{\Omega}$. For an analytic function f of $\{x\} \cup \widetilde{\Omega} = \{x_1, \ldots, x_m\} \cup \widetilde{\Omega}$ let $\text{HD}f$ denote the greatest derivation with respect the occurring in f. For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$, let $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Definition 4.1 *A positive ranking* $\prec$ *of* $\widetilde{\Omega}$ *is a total ordering on* $\widetilde{\Omega}$ *which is compatible with differentiation and well-ordering:*

$$\widetilde{\partial}^\alpha u^i \prec \widetilde{\partial}^\beta u^j \Rightarrow \text{HD}\widetilde{D}^\gamma\widetilde{\partial}^\alpha u^i \prec \text{HD}\widetilde{D}^\gamma\widetilde{\partial}^\beta u^j \tag{17}$$

$$\widetilde{\partial}^\alpha u^i \prec \text{HD}\widetilde{D}^\gamma\widetilde{\partial}^\alpha u^i \textit{ for } |\gamma| \neq 0. \tag{18}$$

Throughout this paper a positive ranking $\prec$ is fixed. Moreover, we suppose that $\prec$ is *compatible with the total degree ordering* that is:

$$|\alpha| < |\beta| \implies \widetilde{\partial}^{\alpha} u^i \prec \widetilde{\partial}^{\beta} u^j \text{ for any } 1 \leq i, j \leq n \tag{19}$$

Thanks to conditions (19) and (13), we have the following property:

$$\text{If } \widetilde{\partial}^{\alpha} u^i \text{ is the highest derivative of } f, \text{ then } \mathrm{HD}(\widetilde{D}^{\beta} f) = \widetilde{\partial}^{\alpha+\beta} u^i \tag{20}$$

There obviously exist positive rankings satisfying (19) such as:

$$\begin{aligned}
\widetilde{\partial}^{\alpha} u^i \prec \widetilde{\partial}^{\beta} u^j \iff & |\alpha| < |\beta|, \text{ or} \\
& |\alpha| = |\beta|, \text{ and } i < j, \text{ or} \\
& |\alpha| = |\beta|, \ i = j \text{ and } \alpha_1 < \beta_1, \text{ or} \\
& |\alpha| = |\beta|, \ i = j, \ \alpha_1 = \beta_1, \cdots, \alpha_{k-1} = \beta_{k-1}, \\
& \qquad \text{and } \alpha_k < \beta_k \text{ for some } 2 \leq k \leq m-1.
\end{aligned}$$

As a consequence on our example this ranking implies:

$$\begin{aligned}
\theta^1 \prec & \theta^2 \prec \theta^3 \prec \widetilde{\partial}_1 \theta^1 \prec \widetilde{\partial}_2 \theta^1 \prec \widetilde{\partial}_3 \theta^1 \prec \widetilde{\partial}_1 \theta^2 \prec \widetilde{\partial}_2 \theta^2 \prec \widetilde{\partial}_3 \theta^2 \prec \widetilde{\partial}_1 \theta^3 \prec \widetilde{\partial}_2 \theta^3 \prec \widetilde{\partial}_3 \theta^3 \\
& \prec \widetilde{\partial}_1 \widetilde{\partial}_1 \theta^1 \prec \widetilde{\partial}_2 \widetilde{\partial}_1 \theta^1 \prec \widetilde{\partial}_2 \widetilde{\partial}_2 \theta^1 \prec \ldots
\end{aligned}$$

According to this ranking, the highest derivation in each equation of (10) with its 5-th equation replaced with (16) can be determined. Solving each equation for its highest derivative with respect to the above ranking yields the system:

$$\begin{array}{lll}
\widetilde{\partial}_3 \theta^1 = -\frac{1}{2} \theta^1 & \widetilde{\partial}_1 \theta^2 = 0 & \widetilde{\partial}_1 \widetilde{\partial}_1 \theta^3 = \widetilde{\partial}_2 \theta^3 \\
& \widetilde{\partial}_2 \theta^2 = 2 \widetilde{\partial}_1 \theta^1 - \theta^3 & \widetilde{\partial}_3 \widetilde{\partial}_1 \theta^3 = \frac{1}{2} \widetilde{\partial}_1 \widetilde{\partial}_1 \theta^1 + (J - \frac{1}{2}) \widetilde{\partial}_1 \theta^3 - \frac{1}{2} \widetilde{\partial}_2 \theta^1 \\
& \widetilde{\partial}_3 \theta^2 = 0 & \widetilde{\partial}_3 \widetilde{\partial}_3 \theta^3 = J \widetilde{\partial}_3 \theta^3 + (\widetilde{\partial}_3 J) \theta^3
\end{array} \tag{21}$$

To check the conditions for our existence and uniqueness theorem for such systems in solved form, we need to determine if certain integrability conditions are satisfied, or reduced to zero modulo the system. Hence in the next section we define and study a suitable reduction process.

5 Reduction

Let f be an analytic function of $\{x\} \cup \widetilde{\Omega}$. We say that f is $\prec$-*monic* if f has the form $f = \mathrm{HD} f + g$, with $\mathrm{HD} g \prec \mathrm{HD} f$. For example the system (21) above is $\prec$-monic.

In the remainder of the paper, let a finite set $\mathcal{M}$ of $\prec$-monic analytic functions of $\{x\} \cup \widetilde{\Omega}$ be fixed. (Other restrictions will be made on $\mathcal{M}$ in section 6).

For g, h two analytic functions of $\{x\} \cup \widetilde{\Omega}$, we say that h is a *one step reduction* of g if there exist $f \in \mathcal{M}$ and $\alpha \in \mathbb{N}^m$ such that, with $\tilde{v}^* := \text{HD}\widetilde{D}^\alpha f$, h can be given by substituting $\tilde{v}^* - \widetilde{D}^\alpha f$ for $\tilde{v}^*$ in g:

$$h = g(x, (\tilde{v})_{\tilde{v} \neq \tilde{v}^*}, (\tilde{v}^* - \widetilde{D}^\alpha f)_{\tilde{v}=\tilde{v}^*}).$$

This is denoted $g \mapsto^{(\alpha,f)} h$, or simply $g \mapsto h$.

We say that g *reduces to* h if h can be obtained from g by a finite chain of one step reductions. That is, g reduces to h if there exists a positive integer k and k functions $h_1, \cdots, h_k$ of $\{x\} \cup \widetilde{\Omega}$ such that

$$g = h_1 \mapsto h_2 \mapsto \cdots \mapsto h_k = h.$$

We write $g \mapsto^\mu h$ or $g \mapsto h$, where μ is of the form

$$\mu = ((\alpha_1, f_1), \cdots, (\alpha_{k-1}, f_{k-1}))$$

with $h_i \mapsto^{(\alpha_i, f_i)} h_{i+1}$. We also write $h = red(g, \mu)$.

We say that g *completely reduces* to h if g reduces to h and h reduces to h' implies that $h = h'$.

Remark 5.1 *The complete reduction may not be unique since may exist two different functions h and $\overline{h}$ such that g completely reduces to both h and $\overline{h}$.*

Example 5.2 *As a consequence of the system (21) the following integrability condition is satisfied $\widetilde{D}_2(\tilde{\partial}_3\theta^2) - \widetilde{D}_3(\tilde{\partial}_2\theta^2) = \widetilde{D}_2(0) - \widetilde{D}_3(2\tilde{\partial}_1\theta^1 - \theta^3)$. Normalization of this equation using commutation relations implies that $-\widetilde{D}_3(2\tilde{\partial}_1\theta^1 - \theta^3) = 0$. Reduction of this last equation with respect to $\tilde{\partial}_3\theta^1 = -\frac{1}{2}\theta^1$ and use of the normalization yields $\tilde{\partial}_3\theta^3 = 0$. Using this relation to reduce $\tilde{\partial}_3\tilde{\partial}_3\theta^3 = J\tilde{\partial}_3\theta^3 + (\tilde{\partial}_3 J)\theta^3$ yields $(\tilde{\partial}_3 J)\theta^3 = 0$ and in summary we have obtained the equations*

$$\tilde{\partial}_3\theta^3 = 0\ , \quad (\tilde{\partial}_3 J)\theta^3 = 0\ . \tag{22}$$

The ad hoc simplification achieved here is only given as an illustration of how reduction can be used to uncover hidden relations from a system. Determination of all the hidden relations, awaits the full development of the theory in the next few sections.

6 Parametric Derivations, Principal Derivations and Non-commutative Riquier Bases

Recall that $\mathcal{M}$ is a finite set of $\prec$-monic analytic functions of $\{x\} \cup \widetilde{\Omega}$. As usual all derivations are assumed to be normalized.

The *principal derivations* of $\mathcal{M}$ are defined as

$$\mathrm{Prin}\mathcal{M} := \{\tilde{v} \in \widetilde{\Omega} | \text{ there exist } f \in \mathcal{M} \text{ and } \alpha \in \mathbb{N}^m \text{ with } \tilde{v} = \mathrm{HD}\widetilde{D}^{\alpha} f\}$$

The *parametric derivations* of $\mathcal{M}$, which we denote $\mathrm{Par}\mathcal{M}$, are those derivations that are not principal.

All leading derivations of elements in $\mathcal{M}$ are in $\mathrm{Prin}\mathcal{M}$, and it is easily shown that $\mathrm{Prin}\mathcal{M}$ are elements of $\widetilde{\Omega}$ which contain some highest derivation as a factor. Therefore a reduction h of g is a complete reduction if and only if h depends on $\{x\} \cup \mathrm{Par}\mathcal{M}$ only.

In this paper, fix a non-empty open subset U of $\mathbb{F}^{\{x\} \cup \widetilde{\Omega}}$. Moreover, we now assume that $\mathcal{M}$ is a set of $\prec$-monic analytic functions which are polynomials in $Prin\mathcal{M}$.

Lemma 6.1 *Let $f, f' \in \mathcal{M}$ and g be an analytic function on U that is a polynomial in $Prin\mathcal{M}$. If there exist non-empty one step reductions: $h = red(g, (\alpha, f))$, $k = red(g, (\beta, f'))$ and $\mathrm{HD}\widetilde{D}^{\alpha} f = \mathrm{HD}\widetilde{D}^{\beta} f'$, then:*

(1) if $\mathrm{HD}\widetilde{D}^{\alpha} f \prec \mathrm{HD}\widetilde{D}^{\beta} f'$ then $red(h, ((\beta, f'), (\alpha, f))) = red(k, (\alpha, f))$.
(2) if $\widetilde{D}^{\alpha} f - \widetilde{D}^{\beta} f' \to^{\mu} 0$, then $red(h, \mu) = red(k, \mu)$

In both cases, there exists an analytic function l such that $h \to l$ and $k \to l$.

PROOF. Let $\tilde{v}^* = \mathrm{HD}\widetilde{D}^{\alpha} f$ and $\tilde{v}^{**} = \mathrm{HD}\widetilde{D}^{\beta} f'$. If $\tilde{v}^* \prec \tilde{v}^{**}$, we have

$$\begin{aligned} red(k, (\alpha, f)) &= g(x, (\tilde{v})_{\tilde{v} \neq \tilde{v}^*, \tilde{v}^{**}}, (\tilde{v}^* - \widetilde{D}^{\alpha} f)_{\tilde{v} = \tilde{v}^*}, (\tilde{v}^{**} - \widetilde{D}^{\beta} f'(x, (\tilde{v})_{\tilde{v} \neq \tilde{v}^*}, \\ &\qquad (\tilde{v}^* - \widetilde{D}^{\alpha} f)_{\tilde{v} = \tilde{v}^*})_{\tilde{v} = \tilde{v}^{**}}) \\ &= red(h, ((\beta, f'), (\alpha, f))) \end{aligned}$$

If $\tilde{v}^* = \tilde{v}^{**}$, then

$$\begin{aligned} red(h,\mu) &= g(x, (red(\tilde{v},\mu))_{\tilde{v}\neq\tilde{v}^*}, (red(\tilde{v} - \widetilde{D}^\alpha f, \mu))_{\tilde{v}=\tilde{v}^*}) \\ &= g(x, (red(\tilde{v},\mu))_{\tilde{v}\neq\tilde{v}^*}, (red(\tilde{v},\mu) - red(\widetilde{D}^\alpha f, \mu))_{\tilde{v}=\tilde{v}^*}) \\ &= g(x, (red(\tilde{v},\mu))_{\tilde{v}\neq\tilde{v}^*}, (red(\tilde{v},\mu) - red(\widetilde{D}^\beta f', \mu))_{\tilde{v}=\tilde{v}^*}) \\ &= red(k,\mu). \end{aligned}$$

□

Lemma 6.2 (Diamond Lemma) *Fix $\tilde{v} \in \widetilde{\Omega}$. Suppose that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with $\mathrm{HD}\widetilde{D}^\alpha f = \mathrm{HD}\widetilde{D}^{\alpha'} f' \leq \tilde{v}$, we have $\widetilde{D}^\alpha f - \widetilde{D}^{\alpha'} f' \to 0$. Let g be an analytic function on U that is polynomial in $Prin\mathcal{M}$ with $\mathrm{HD}g \leq \tilde{v}$, and two non-empty reductions $g \to h, g \to k$. Then there exists l with $h \to l$ and $k \to l$. In particular, g has an unique complete reduction.*

PROOF. The proof is very similar to the proof of the uniqueness of the normal form of a polynomial modulo a Gröbner basis (for example, see [3]). Also see [30, page 67] where it is given in the commutative case. □

The use of the bound $\tilde{v}$ on the highest derivative is needed later in the proofs of Lemma 8.1 and Theorem 8.5.

Definition 6.3 *$\mathcal{M}$ is called a non-commutative Riquier Basis if for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with $\mathrm{HD}\widetilde{D}^\alpha f = \mathrm{HD}\widetilde{D}^{\alpha'} f'$, the integrability condition $\widetilde{D}^\alpha f - \widetilde{D}^{\alpha'} f' \to 0$.*

From above lemmas, it is easy to see that:

Theorem 6.4 *Suppose that $\mathcal{M}$ is a non-commutative Riquier Basis and g is an analytic function on U that is polynomial in $Prin\mathcal{M}$. Then g has an unique complete reduction.*

We denote the complete reduction of g by $red(g, \mathcal{M})$.

7 The Formal Non-commutative Riquier Existence Theorem

Let f be an $\mathbb{F}$-analytic function of $\{x\} \cup \widetilde{\Omega}$, $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$, and x^0 be a point in $\mathbb{F}^m$ and let $u(x) = (u^1(x), \cdots, u^n(x))$ be a vector of formal power series in $\mathbb{F}[[x - x^0]]^n$.

If f is defined at the point $(x^0, (\widetilde{D}^\alpha u^i(x^0))_{\tilde{v}^i_\alpha \in \widetilde{\Omega}})$, let $f[u](x)$ denote the formal power series at x^0 given by

$$f[u](x) := f(x, ((\widetilde{D}^\alpha u^i(x))_{\tilde{v}^i_\alpha \in \widetilde{\Omega}}).$$

where the subscript "$\tilde{v}^i_\alpha \in \widetilde{\Omega}$" indicates that $\widetilde{D}^\alpha u^i(x)$ is to be substituted in the argument of f corresponding to $\tilde{v}^i_\alpha$ for each $\tilde{v}^i_\alpha \in \widetilde{\Omega}$.

We illustrate these concepts with a simple example.

Example 7.1 *Let $m = n = 1$ and $x^0 = 1$. Here u^1, x_1, $\widetilde{\partial}_1$ and ∂_1 are simply denoted u, x, $\widetilde{\partial}$ and ∂. The relation (12) is simply denoted $\widetilde{\partial} = a(x)\partial$.*

Let $u(x)$ be the formal power series

$$u(x) = 1 + (x-1) + 2!(x-1)^2 + \cdots = \sum_{k=0}^{\infty} k!(x-1)^k$$

and let $f = \ln(\tilde{v}^1_{(1)})$ (recall that $\tilde{v}^1_{(1)}[u](x) = \widetilde{\partial}u(x)$).

For $k \geq 1$, we have $\widetilde{\partial}((x-1)^k) = a(x)k(x-1)^{k-1}$. Differentiating $u(x)$ term by term (which is the definition of the derivative of a formal power series) we obtain

$$\tilde{v}^1_{(1)}[u](x) = \widetilde{\partial}(u(x)) = a(x)\sum_{k=1}^{\infty}(k+1)\,(k+1)!\,(x-1)^k$$

Note that the $\ln(y)$ function is analytic at the constant term of the series $u(x)$, i.e. at the point $y = 1$. Thus, the series $f[u](x)$ is well defined and equals:

$$\begin{aligned} f[u](x) &= \ln(\tilde{v}^1_{(1)}[u](x)) \\ &= -\textstyle\sum_{j=1}^{\infty} \frac{(-1)^j}{j}(\tilde{v}^1_{(1)}[u](x) - 1)^j \\ &= -a(x)\textstyle\sum_{j=1}^{\infty} \frac{(-1)^j}{j}\left(\sum_{k=1}^{\infty}(k+1)(k+1)!(x-1)^k\right)^j \end{aligned}$$

We say that $u(x) \in \mathbb{F}[[x - x^0]]^n$ (for some $x^0 \in \mathbb{F}^m$) is a *formal power series solution* to a system of analytic PDE if $f[u](x)$ is well-defined and $f[u](x) = 0$ for all f in the system.

Suppose that $u(x) \in \mathbb{F}[[x - x^0]]^n$ is a formal power series solution to $\mathcal{M}$. Clearly, $\widetilde{D}^\alpha f[u](x) = 0$ for all $\alpha \in \mathbb{N}^m$ and $f \in \mathcal{M}$. Therefore for g, h analytic, if h is a one step reduction of g then $h[u](x)$ is well-defined if and only if $g[u](x)$ is well-defined and in this case $g[u](x) = h[u](x)$. Furthermore, u is a formal power series solution to $\mathcal{M} \cup \{g\}$ iff u is a formal power series solution to $\mathcal{M} \cup \{h\}$.

A *specification of initial data* for $\mathcal{M}$ is a map

$$\phi : \{x\} \cup \text{ Par } \mathcal{M} \to \mathbb{F}$$

For $x^0 \in \mathbb{F}^m$, we say that ϕ is a specification at x^0 if

$$\phi(x) := (\phi(x_1), \phi(x_2), \cdots, \phi(x_m)) = x^0.$$

For g a function of $\{x\} \cup \widetilde{\Omega}$, let $\phi(g)$ be the function of the principal derivations obtained from g by evaluating x and the parametric derivations using ϕ:

$$\phi(g) := g(\phi(x), (\phi(\tilde{v}))_{\tilde{v} \in Par\mathcal{M}}).$$

Lemma 7.2 (Uniform Reduction) *let $\mathcal{G} := \{g_1, g_2, \cdots, g_k\}$ be a finite set of functions of $\{x\} \cup \widetilde{\Omega}$. Then there exists μ such that $red(g, \mu)$ is a complete reduction of g for all $g \in \mathcal{G}$.*

PROOF. By Dickson's lemma complete reductions always exist and we can choose μ_1 such that $red(g_1, \mu_1)$ is a complete reduction of g_1. Recursively construct $red(g_j, (\mu_1, \mu_2, \cdots, \mu_j))$ which for $j = 2, \cdots, k$, is a complete reduction of $red(g_j, (\mu_1, \cdots, \mu_{j-1}))$ and hence a complete reduction of g_j. Set $\mu = (\mu_1, \cdots, \mu_k)$. Thus for $j \in \{1, \cdots, k\}$ we have

$$red(g_j, \mu) = red(red(g_j, (\mu_1, \cdots, \mu_j)), (\mu_{j+1}, \cdots, \mu_k)) = red(g_j, (\mu_1, \cdots, \mu_j)),$$

which is a complete reduction of g_j by construction. □

Corollary 7.3 *If $\mathcal{M}$ is a non-commutative Riquier Basis and $\mathcal{G}$ is a finite set of functions of $\{x\} \cup \widetilde{\Omega}$, then there exists μ such that $red(g, \mu) = red(g, \mathcal{M})$ for all $g \in \mathcal{G}$.*

Theorem 7.4 (Formal Non-commutative Riquier Existence Theorem) *Let $\mathcal{M}$ be a non-commutative Riquier Basis such that each $f \in \mathcal{M}$ is polynomial in the principal derivations (e.g. $\mathcal{M}$ is a reduced non-commutative Riquier basis). For $x^0 \in \mathbb{F}^m$, let ϕ be a specification of initial data for $\mathcal{M}$ at x^0 such that $\phi(f)$ is well-defined for all $f \in \mathcal{M}$. Then there is an unique formal power series solution $u(x) \in \mathbb{F}[[x - x^0]]^n$ to $\mathcal{M}$ at x^0 such that $\widetilde{D}^\alpha u^i(x^0) = \phi(\tilde{v}^i_\alpha)$ for all $\tilde{v}^i_\alpha \in Par\mathcal{M}$. Furthermore, every formal power series solution to $\mathcal{M}$ at x^0 may be obtained in this way for some ϕ.*

PROOF. By the bijective correspondence of Theorem 3.4, there exists a n-uple formal power series $u(x) \in \mathbb{F}[[x - x^0]]^n$ satisfying

$$\widetilde{D}^\alpha u^i(x^0) := \phi(red(\tilde{v}^i_\alpha, \mathcal{M})) \qquad (23)$$

for $i \in \{1, \cdots, m\}$.

Since $u(x)$ must satisfy equation (23) for all $i \in \{1, \cdots, n\}$ and $\alpha \in \mathbb{N}^m$ and since by Theorem 3.4 we have a bijection between derivations and partial derivatives the formal power series solution (if it exists) is unique.

We now prove that $u(x)$ is a formal power series solution of the system, which will prove the existence part of the theorem.

(1) We have first to check that $\phi(red(\tilde{v}^i_\alpha, \mathcal{M}))$ is well-defined.

Note that $\phi(red(\tilde{v}^i_\alpha, \mathcal{M}))$ depends only on the parametric derivations and so it is an element of $\mathbb{F}$, so long as it is well-defined.

(2) Then we have to verify that $u(x)$ is a formal power series solution to $\mathcal{M}$.

Clearly, $u(x)$ satisfies $\widetilde{D}^\alpha u^i(x^0) = \phi(\tilde{v}^i_\alpha)$ for all $\tilde{v}^i_\alpha \in Par\mathcal{M}$. Now it suffices to verify that $\widetilde{D}^\beta f[u](x^0) = 0$ for all $f \in \mathcal{M}$ and $\beta \in \mathbb{N}^m$. Hypothesis 3.2 will imply $D^\beta f[u](x^0) = 0$ for all $f \in \mathcal{M}$ and $\beta \in \mathbb{N}^m$ and consequently that $f[u]$ is the zero formal power series.

Fix f, β. We have

$$\begin{aligned}\widetilde{D}^\beta f[u](x^0) &= (\widetilde{D}^\beta f)(x^0, (\widetilde{D}^\beta u^i(x^0))) \\ &= (\widetilde{D}^\beta f)(\phi(x^0), (\phi(red(\tilde{v}^i_\alpha, \mathcal{M})))) \\ &= \phi(\widetilde{D}^\beta f(x, (red(\tilde{v}^i_\alpha, \mathcal{M}))))\end{aligned}$$

Let $\widetilde{\Omega}'$ be the finite subset of $\widetilde{\Omega}$ on which $\widetilde{D}^\beta f$ depends. By Lemma 7.2 there exists μ such that for all $\tilde{v} \in \widetilde{\Omega}'$

$$red(\tilde{v}^i_\alpha, \mathcal{M}) = red(\tilde{v}^i_\alpha, \mu).$$

Therefore

$$\begin{aligned}\widetilde{D}^\beta f[u](x^0) &= \phi(\widetilde{D}^\beta f(x, (red(\tilde{v}^i_\alpha, \mu)))) \\ &= \phi(red(\widetilde{D}^\beta f(x, \tilde{v}^i_\alpha), \mu)).\end{aligned}$$

Note that $red(\widetilde{D}^\beta f(x, \tilde{v}^i_\alpha), \mu)$ depends only on the parametric derivations and x. Hence it is a complete reduction of $\widetilde{D}^\beta f(x, \tilde{v}^i_\alpha)$ and we have

$$\begin{aligned}red(\widetilde{D}^\beta f(x, \tilde{v}^i_\alpha), \mu) &= red(\widetilde{D}^\beta f(x, \tilde{v}^i_\alpha), \mathcal{M})) \\ &= red(\widetilde{D}^\beta f(x, \tilde{v}^i_\alpha), (\beta, f)) \\ &= 0\end{aligned}$$

Thus $(\widetilde{D}^{\beta} f[u])(x^0) = 0$ as required.

This completes the proof of existence part of the theorem.

□

8 Sufficient Finite Sets of Integrability Conditions

Note that the Formal Non-commutative Riquier Existence Theorem 7.4 requires the checking of infinitely many integrability conditions. In this section we show that only finitely many integrability conditions need to be checked.

Lemma 8.1 (Reduction of a sum) *Suppose h, k are polynomials in $Prin\mathcal{M}$. Suppose $h \to^{\mu} 0$ and $k \to^{\nu} 0$. Suppose further that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with $\text{HD}\widetilde{D}^{\alpha} f = \text{HD}\widetilde{D}^{\alpha'} f' \leq \text{HD}k$, we have $\widetilde{D}^{\alpha} f - \widetilde{D}^{\alpha'} f' \to 0$. Then we have $h + k \to 0$.*

PROOF. There are two cases:

(1) If $red(k, \mu)$ is an empty reduction, then

$$\begin{aligned} red(h + k, (\mu, \nu)) &= red(h, (\mu, \nu)) + red(k, (\mu, \nu)) \\ &= red(0, \nu) + red(k, \nu) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

(2) If $red(k, \mu)$ is a non-empty reduction, say $l = red(k, \mu)$, then by Lemma 6.2 there exist j, l with $0 \to j$ and $l \to j$. Since $0 \to j$, we have $j = 0$ and hence $l \to 0$, say $0 = red(l, \rho)$. Then we have:

$$\begin{aligned} red(h + k, (\mu, \rho) &= red(h, (\mu, \rho)) + red(k, (\mu, \rho)) \\ &= red(0, \rho) + red(l, \rho) \\ &= 0. \end{aligned}$$

Therefore, $h + k \to 0$, as required.

□

Lemma 8.2 (Reduction of a Derivation) *Take* $\alpha \in \mathbb{N}^m, f \in \mathcal{M}, i \in \{1, \cdots, n\}$ *and* g *an analytic function on* U *that is polynomial in* $Prin\mathcal{M}$*. Let* $\tilde{v}^{**} = {}_{\mathrm{HD}}\widetilde{D}^{\alpha} f$ *and let* $\tilde{v}^{*}$ *be given by* $\tilde{v}^{**} = {}_{\mathrm{HD}}\widetilde{D}_i \tilde{v}^{*}$*, if this is well-defined.*

Then

$$red(\widetilde{D}_i g, (\alpha, f)) = \widetilde{D}_i\, red(g, (\alpha, f)) + red(\tfrac{\partial g}{\partial \tilde{v}^{**}}, (\alpha, f))\widetilde{D}_i \widetilde{D}^{\alpha} f \\ - red(\tfrac{\partial g}{\partial \tilde{v}^{*}}, (\alpha, f))\widetilde{D}^{\alpha} f.$$

If $\tilde{v}^{*}$ *is not well-defined, then the last term is omitted in the above formula.*

PROOF. By the definition of reduction we have:

$$red(g, (\alpha, f)) = g(x, (\tilde{v})_{\tilde{v} \neq \tilde{v}^{**}}, (\tilde{v}^{**} - \widetilde{D}^{\alpha} f)_{\tilde{v} = \tilde{v}^{**}}). \tag{24}$$

Therefore, using equation (15) and using the property that the operations red commute with any ∂_i or $\frac{\partial}{\partial \tilde{v}}$ yields

$$\widetilde{D}_i\ red(g, (\alpha, f)) = red(\tilde{\partial}_i g, (\alpha, f)) + \textstyle\sum_{\tilde{v} \neq \tilde{v}^{**}} red(\tfrac{\partial g}{\partial \tilde{v}}, (\alpha, f))\widetilde{D}_i \tilde{v} \\ +\ red(\tfrac{\partial g}{\partial \tilde{v}^{**}}, (\alpha, f))\widetilde{D}_i(\tilde{v}^{**} - \widetilde{D}^{\alpha} f). \tag{25}$$

Also by the definition of total derivation we have

$$\widetilde{D}_i g = \tilde{\partial}_i g + \sum_{\tilde{v} \in \widetilde{\Omega}} \frac{\partial g}{\partial \tilde{v}} \widetilde{D}_i \tilde{v}.$$

Thus

$$red(\widetilde{D}_i g, (\alpha, f)) = red(\tilde{\partial}_i g, (\alpha, f)) + \textstyle\sum_{\tilde{v} \neq \tilde{v}^{*}} red(\tfrac{\partial g}{\partial \tilde{v}}, (\alpha, f))\widetilde{D}_i \tilde{v} \\ +\ red(\tfrac{\partial g}{\partial \tilde{v}^{*}}, (\alpha, f))\, red(\widetilde{D}_i \tilde{v}^{*}, (\alpha, f)) \tag{26}$$

The proof is continued on the next page.

Solving (25) for $red(\widetilde{\partial}_i g, (\alpha, f))$, and then eliminating this from (26) yields

$$
\begin{aligned}
red(\widetilde{D}_i g, (\alpha, f)) &= \widetilde{D}_i\, red(g, (\alpha, f)) - red(\tfrac{\partial g}{\partial \tilde{v}^{**}}, (\alpha, f))\widetilde{D}_i(\tilde{v}^{**} - \widetilde{D}^\alpha f) \\
&\quad - \textstyle\sum_{\tilde{v} \neq \tilde{v}^{**}} red(\tfrac{\partial g}{\partial \tilde{v}}, (\alpha, f))\widetilde{D}_i \tilde{v} \\
&\quad + \textstyle\sum_{\tilde{v} \neq \tilde{v}^{*}} red(\tfrac{\partial g}{\partial \tilde{v}}, (\alpha, f))\widetilde{D}_i \tilde{v} + \; red(\tfrac{\partial g}{\partial \tilde{v}^{*}}, (\alpha, f))\, red(\widetilde{D}_i \tilde{v}^*, (\alpha, f)) \\
&= \widetilde{D}_i\, red(g, (\alpha, f)) - red(\tfrac{\partial g}{\partial \tilde{v}^{**}}, (\alpha, f))\widetilde{D}_i(\tilde{v}^{**} - \widetilde{D}^\alpha f) \\
&\quad - \textstyle\sum_{\tilde{v} \neq \tilde{v}^{**}, \tilde{v}^*} red(\tfrac{\partial g}{\partial \tilde{v}}, (\alpha, f))\widetilde{D}_i \tilde{v} - red(\tfrac{\partial g}{\partial \tilde{v}^{*}}, (\alpha, f))\widetilde{D}_i \tilde{v}^* \\
&\quad + \textstyle\sum_{\tilde{v} \neq \tilde{v}^{*}, \tilde{v}^{**}} red(\tfrac{\partial g}{\partial \tilde{v}}, (\alpha, f))\widetilde{D}_i \tilde{v} + red(\tfrac{\partial g}{\partial \tilde{v}^{**}}, (\alpha, f))\widetilde{D}_i \tilde{v}^{**} \\
&\quad + red(\tfrac{\partial g}{\partial \tilde{v}^{*}}, (\alpha, f))\, red(\widetilde{D}_i \tilde{v}^*, (\alpha, f)) \\
&= \widetilde{D}_i\, red(g, (\alpha, f)) + red(\tfrac{\partial g}{\partial \tilde{v}^{**}}, (\alpha, f))\widetilde{D}_i \widetilde{D}^\alpha f + \\
&\quad red(\tfrac{\partial g}{\partial \tilde{v}^{*}}, (\alpha, f))[red(\widetilde{D}_i \tilde{v}^*, (\alpha, f)) - \widetilde{D}_i \tilde{v}^*]
\end{aligned}
\tag{27}
$$

Since the term $\widetilde{D}_i \tilde{v}^*$ is not normalized, we have to be careful before applying the red operation. We can write $\widetilde{D}_i \tilde{v}^* = \tilde{v}^{**} + \sum_\nu a_\nu \tilde{v}_\nu$ where the sum is finite, the a_ν's analytic functions and the $\tilde{v}_\nu$ belong to $\widetilde{\Omega}$ and are different from $\tilde{v}^{**}$. Thus we have

$$
\begin{aligned}
red(\widetilde{D}_i \tilde{v}^*, (\alpha, f)) - \widetilde{D}_i \tilde{v}^* &= red(\tilde{v}^{**} + \textstyle\sum_\nu a_\nu \tilde{v}_\nu, (\alpha, f)) - (\tilde{v}^{**} + \sum_\nu a_\nu \tilde{v}_\nu) \\
&= red(\tilde{v}^{**}, (\alpha, f)) - \tilde{v}^{**} \\
&= (\tilde{v}^{**} - \widetilde{D}^\alpha f) - \tilde{v}^{**} \\
&= -\widetilde{D}^\alpha f
\end{aligned}
$$

Inserting this expression into (27) ends the proof of the lemma. □

Lemma 8.3 *Let g be an analytic function on U such that $g \to 0$ with respect to $\mathcal{M}$. Fix $i \in \{1, \cdots, m\}$. Suppose that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with* $\mathrm{HD}\widetilde{D}^\alpha f = \mathrm{HD}\widetilde{D}^{\alpha'} f' \prec \mathrm{HD}\widetilde{D}_i g, \widetilde{D}^\alpha f - \widetilde{D}^{\alpha'} f' \to 0$*. Then $\widetilde{D}_i g \to 0$.*

PROOF. By the induction on the length of the minimal chain required to reduce g to 0, we may assume that there exists an analytic function $h \neq g$ of

$\{x\} \cup \widetilde{\Omega}$ with $g \to h \to 0$ and $\widetilde{D}_i h \to 0$, say $h = red(g, (\alpha_h, f_\alpha))$. By Lemma 8.2, we have an expression of the form

$$red(\widetilde{D}_i g, (\alpha_h, f_h)) = \widetilde{D}_i h + k\widetilde{D}_i \widetilde{D}^{\alpha_h} f_h + l\widetilde{D}^{\alpha_k} f_h. \tag{28}$$

with k and l analytic functions of $\{x\} \cup \widetilde{\Omega}$ satisfying $\textsc{hd}k \prec \textsc{hd}g$ and $\textsc{hd}l \prec \textsc{hd}g$. Furthermore, either $\textsc{hd}\widetilde{D}_i h \prec \textsc{hd}\widetilde{D}_i g$ or $\textsc{hd}\widetilde{D}_i \widetilde{D}^\alpha f \prec \textsc{hd}\widetilde{D}_i g$. In any case, at least two of three summands in above equation have highest derivative strictly less than $\textsc{hd}\widetilde{D}_i g$. Therefore by two applications of Lemma 8.1, we have $red(\widetilde{D}_i g, (\alpha_h, f_h)) \to 0$ and so $\widetilde{D}_i g \to 0$.

□

The *least common multiple* of $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\alpha' = (\alpha'_1, \ldots, \alpha'_m)$ is defined by $(\max(\alpha_1, \alpha'_1), \ldots, \max(\alpha_m, \alpha'_m))$.

Definition 8.4 *Let $f, f' \in \mathcal{M}$ with $\textsc{hd}f = \widetilde{D}^\alpha u^i$ and $\textsc{hd}f' = \widetilde{D}^{\alpha'} u^{i'}$, and β be the least common multiple of α and α'. Then if $i = i'$, define the minimal integrability condition of f and f' to be $\textsc{ic}(f, f') = \widetilde{D}^{\beta-\alpha} f - \widetilde{D}^{\beta-\alpha'} f'$. If $i \neq i'$, then $\textsc{ic}(f, f')$ is said to be undefined.*

Theorem 8.5 *Suppose that for each pair $f, f' \in \mathcal{M}$ with $\textsc{ic}(f, f')$ well-defined we have $\textsc{ic}(f, f') \to 0$. Then $\mathcal{M}$ is a non-commutative Riquier Basis.*

PROOF. Take $f, f' \in \mathcal{M}$ and $\alpha, \alpha' \in \mathbb{N}^m$ such that $\textsc{hd}\widetilde{D}^\alpha f = \textsc{hd}\widetilde{D}^{\alpha'} f'$. We have to show that $\widetilde{D}^\alpha f - \widetilde{D}^{\alpha'} f' \to 0$. We proceed by induction on the highest derivation in $\widetilde{D}^\alpha f$.

The basis for the induction is ensured by the assumption $\textsc{ic}(f, f') \to 0$.

Now assume that $\widetilde{D}^{\alpha^*} f^* - \widetilde{D}^{\alpha^{**}} f^{**} \to 0$ for $f^*, f^{**} \in \mathcal{M}$ and $\alpha^*, \alpha^{**} \in \mathbb{N}^m$ with $\textsc{hd}\widetilde{D}^{\alpha^*} f^* = \textsc{hd}\widetilde{D}^{\alpha^{**}} f^{**} \prec \textsc{hd}\widetilde{D}^\alpha f$.

Suppose that $\widetilde{D}^\alpha f - \widetilde{D}^{\alpha'} f'$ is not equal to $\textsc{ic}(f, f')$. Thus there exist γ, β and β' in $\mathbb{N}_n$ such that $\gamma \neq (0, \ldots, 0)$, $\alpha = \gamma + \beta$, $\alpha' = \gamma + \beta'$ and $\widetilde{D}^\beta f - \widetilde{D}^{\beta'} f' = \textsc{ic}(f, f')$.

In contrast to the commutative case, we do not have $\widetilde{D}^\alpha f - \widetilde{D}^{\alpha'} f' = \widetilde{D}^\gamma(\textsc{ic}(f, f'))$ for some γ. However, using the commutation rules, we have the following relation:

$$\widetilde{D}^\alpha f - \widetilde{D}^{\alpha'} f' = \widetilde{D}^\gamma(\textsc{ic}(f, f')) + \sum_\nu a_\nu \widetilde{D}^\nu f + \sum_{\nu'} a_{\nu'} \widetilde{D}^{\nu'} f'$$

where the two sums are finite, a_ν and $a_{\nu'}$ are analytic functions and $\text{HD}\widetilde{D}^{\nu} f \prec \text{HD}\widetilde{D}^{\alpha} f$ and $\text{HD}\widetilde{D}^{\nu'} f' \prec \text{HD}\widetilde{D}^{\alpha} f$.

Using the induction hypothesis and by (repeated) applications of Lemma 8.1, we have $\widetilde{D}^{\alpha} f - \widetilde{D}^{\alpha'} f' \to 0$.

□

Example 8.6 *We return to our running example, the frame treatment of the defining system (5) of PDE for infinitesimal symmetries of the nonlinear Heat equation (3). From (22) for our system (21) we have two cases.*

Case 1: $\theta^3 = 0\ ,\ \ \tilde{\partial}_3 J \neq 0$*;* **Case 2:** $\tilde{\partial}_3 J = 0$*.*

Case 1 ($\theta^3 = 0$, $\tilde{\partial}_3 J \neq 0$)*. Reducing the system with respect to* $\theta^3 = 0$ *yields* $\frac{1}{2}\tilde{\partial}_1\tilde{\partial}_1\theta^1 - \frac{1}{2}\tilde{\partial}_2\theta^1 = 0$*. Computing and reducing the integrability condition between this equation and* $\tilde{\partial}_3\theta^1 = -\frac{1}{2}\theta^1$ *gives* $\tilde{\partial}_1\tilde{\partial}_1\theta^1 = 0$ *and* $\tilde{\partial}_2\theta^1 = 0$*. In summary the system for this case becomes:*

$$\begin{array}{lll} \tilde{\partial}_1\tilde{\partial}_1\theta^1 = 0 & \tilde{\partial}_1\theta^2 = 0 & \theta^3 = 0 \\ \tilde{\partial}_2\theta^1 = 0 & \tilde{\partial}_2\theta^2 = 2\tilde{\partial}_1\theta^1 & \\ \tilde{\partial}_3\theta^1 = -\frac{1}{2}\theta^1 & \tilde{\partial}_3\theta^2 = 0 & \end{array} \tag{29}$$

It can be checked that all the integrability conditions for this system are satisfied and it satisfies all the conditions for a non-commutative Riquier Basis. There are three parametric derivations θ^1, θ^2, $\tilde{\partial}_1\theta^1$*. Hence by the Non-commutative Riquier Existence and Uniqueness Theorem its symmetry algebra is of dimension three.*

Case 2 ($\tilde{\partial}_3 J = 0$)*. Further compatibility conditions and reductions yield the condition* $(3-4J)\tilde{\partial}_1\theta^3 = 0$*. Thus there are two cases:* **Case 2a:** $J \neq \frac{3}{4}, \tilde{\partial}_1\theta^3 = 0$ *and* **Case 2b:** $J = \frac{3}{4}$*.*

Case 2a: ($J \neq \frac{3}{4}, \tilde{\partial}_1\theta^3 = 0$)*. We obtain:*

$$\begin{array}{lll} \tilde{\partial}_1\tilde{\partial}_1\theta^1 = 2(1-J)\tilde{\partial}_1\theta^3 & \tilde{\partial}_1\theta^2 = 0 & \tilde{\partial}_1\tilde{\partial}_1\theta^3 = 0 \\ \tilde{\partial}_2\theta^1 = 0 & \tilde{\partial}_2\theta^2 = 2\tilde{\partial}_1\theta^1 - \theta^3 & \tilde{\partial}_2\theta^3 = 0 \\ \tilde{\partial}_3\theta^1 = -\frac{1}{2}\theta^1 & \tilde{\partial}_3\theta^2 = 0 & \tilde{\partial}_3\theta^3 = 0 \end{array} \tag{30}$$

It can be checked that all the conditions for a non-commutative Riquier Basis are satisfied. There are four parametric derivations θ^1, θ^2, $\tilde{\partial}_1\theta^1$, θ^3*. Hence its symmetry algebra is of dimension four.*

Case 2b: $J = \frac{3}{4}$. *The system becomes the non-commutative Riquier Basis:*

$$\begin{array}{lll}
\widetilde{\partial}_1\widetilde{\partial}_1\theta^1 = 0 & \widetilde{\partial}_1\theta^2 = 0 & \widetilde{\partial}_1\theta^3 = 0 \\
\widetilde{\partial}_2\theta^1 = 0 & \widetilde{\partial}_2\theta^2 = 2\widetilde{\partial}_1\theta^1 - \theta^3 & \widetilde{\partial}_2\theta^3 = 0 \\
\widetilde{\partial}_3\theta^1 = -\frac{1}{2}\theta^1 & \widetilde{\partial}_3\theta^2 = 0 & \widetilde{\partial}_3\theta^3 = 0.
\end{array}$$

There are five parametric derivations θ^1, θ^2, θ^3, $\widetilde{\partial}_1\theta^1$, $\widetilde{\partial}_1\theta^3$, yielding a five-dimensional symmetry algebra.

9 Analyticity Issues

Theorem 7.4 gives existence and uniqueness conditions for a formal power series solution for an associated specification of initial data. The Riquier-Janet Existence and Uniqueness Theorem for the commutative case states that, under certain assumptions, a specification of *analytic* initial data yields an *analytic* power series solution.

In this section, we investigate the generalization of this analyticity theorem to the non-commutative case by seeking conditions on the initial data specification ensuring that the formal power series solution is analytic.

Riquier [29] and Janet [13] consider systems of PDE with commuting derivations. They consider orthonomic and passive systems to express the analyticity theorem. Instead of defining the orthonomic and passive systems, we will use the non-commutative Riquier Basis described in this paper to state the Riquier analyticity theorem. We need the following definitions.

Definition 9.1 *A Riquier ranking $\prec$ is a positive ranking satisfying $\widetilde{\partial}^\alpha u^i \prec \widetilde{\partial}^\beta u^i \iff \widetilde{\partial}^\alpha u^j \prec \widetilde{\partial}^\beta u^j$ for any i and j.*

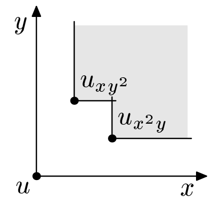
Definition 9.2 *A* specification of initial data ϕ *for a system $\mathcal{M}$ is* analytic *if there exist two positive real numbers M and r such that $|\phi(\widetilde{\partial}^\alpha u^i)| \leq M r^{|\alpha|}\alpha!$ for all $\widetilde{\partial}^\alpha u^i$ in $Par\mathcal{M}$.*

With the above definitions we can state the Riquier analyticity theorem in the following alternative form.

Theorem 9.3 *Let $\prec$ be a Riquier ranking compatible with the total degree ordering. Suppose that the $\widetilde{\partial}_i$ commute. Consider a non-commutative Riquier Basis $\mathcal{M}$ and analytic initial data specification ϕ. Then the unique formal power series solution thus defined is also analytic.*

In the Riquier-Janet Theory, specifying the initial conditions is equivalent to fixing the values of the dependant variables and their derivatives at a point x^0.

Example 9.4 $\begin{cases} u_{xxy} = f(u) \\ u_{xyy} = g(u) \end{cases}$



Choosing a specification of initial data around the origin $x = 0, y = 0$ in the Riquier-Janet approach amounts to fixing the value of u_{xy} at $x = y = 0$, and fixing the value of u on $\{x = 0\} \cup \{y = 0\}$. A precise contruction of the initial data based on multiplicative variables [13] would consist in fixing the value of u_{xy} on $M_0 = \{(0, 0)\}$; fixing the value u on $M_1 = \{(x, 0) : x \in \mathbb{R}\}$; and fixing the value u_y on $M_2 = \{(0, y) : y \in \mathbb{R}\}$.

More generally, a specification of initial data can be expressed in the following geometric manner, which is more suitable for our investigation of the non-commutative case. Choosing a specification of initial data around x^0 is equivalent to assigning functions to some parametric derivatives along specific sub-manifolds M_i. The choice of the parametric derivatives and the M_i is described in [13]. Moreover, choosing an analytic initial data specification is equivalent to assigning analytic functions to the dependant variables on the sub-manifolds M_i.

Extending these results to the non-commutative case leads to the following questions:

- What is the geometric meaning of fixing an initial data specification?
- What criteria must the initial data satisfy to ensure the analyticity of the associated formal power series solution?

We first consider the case where there are a finite number of parameters in the formal power series solution.

Theorem 9.5 (Analyticity in the finite parameter case) *Let $\prec$ be a Riquier ranking compatible with the total degree ordering. Consider a non-commutative Riquier Basis $\mathcal{M}$ with a finite set of parametric derivations $Par\mathcal{M} = \{w^1, ..., w^k\}$. Then the formal power series solution about x^0 with initial data $w^1(x^0) = w^1_0, ..., w^k(x^0) = w^k_0$ is analytic at x^0.*

PROOF. For $i = 1, ..., m$ any $\widetilde{D}_i w^l \in \mathrm{Prin}\mathcal{M}$ can be completely reduced by $\mathcal{M}$ to an analytic function f^l_i of $\{x\} \cup \mathrm{Par}\mathcal{M}$ such that

$$\widetilde{D}_i w^l = f^l_i. \tag{31}$$

Now from (14) it follows that (31) is equivalent to

$$D_i w^l = \sum_j b_{ij}(x,u) f_j^l \ , \tag{32}$$

where $b(x,u)$ is the inverse matrix of $a(x,u)$.

The easily computed integrability conditions of (32) are analytic functions of $\{x\} \cup \mathrm{Par}\mathcal{M}$. If one integrability condition was not satisfied, there would a set of initial conditions for (32) such that (32) does not admit a solution. Thus, for the initial conditions, (31) and $\mathcal{M}$ would not admit a solution, which contradicts the existence theorem of a solution for the non-commutative Riquier Basis $\mathcal{M}$. Thus the system (32) is a commutative Riquier Basis, and by the standard commutative theory, must have a formal power series solution with the given data, which is analytic at x^0. □

We also have:

Theorem 9.6 *Under the hypotheses of Theorem 9.5, the integration of $\mathcal{M}$ is equivalent to integrating a system of ODE along an analytic curve.*

PROOF. Consider an analytic curve $x(\tau) = x_i(\tau)$, with $x(0) = x^0$. Then $\frac{dw^l}{d\tau} = \sum_i \frac{dx_i}{d\tau}\frac{\partial w^l}{\partial x_i}$ which from (32) yields the system of ODE:

$$\frac{dw^l}{d\tau} = \sum_i \frac{dx_i}{d\tau} \sum_j b_{ij}(x,u) f_j^l \ . \tag{33}$$

□

This gives an answer to a question posed by Mansfield (private communication), about the ways in which the output of non-commutative differential elimination methods can be used. Traditional commutative differential elimination packages often use elimination rankings to decouple ODE which can then be sometimes exactly integrated by ODE solvers. Theorem 9.6 gives an alternative method for exposing ODE systems. It is interesting to explore to what extent geometric ODE integrators (numerical integrators invariant under the admitted Lie group), could be fruitfully applied to such systems using Theorem 9.6. Calculating the induced group action on the ODE system above, and exploring to what extent invariant descriptions can be found by suitably choosing the parameterization and curves $x(\tau)$ are interesting open problems.

We now consider the case of a non-commutative Riquier Basis $\mathcal{M}$ where $\mathrm{Par}\mathcal{M}$ is infinite. In the commutative case, the geometric theory of PDE [25], the geometric prolongation of the system to an order r is obtained by applying

D_i to the equations of the system until no undifferentiated equations of order r or less remain. Equivalently one may obtain the geometric prolongation by similarly using $\widetilde{D}_i$. Since the system is a non-commutative Riquier Basis any prolongation of the system is formally integrable (as defined in the geometric theory). It is also a consequence of the geometric theory that some finite order prolongation of the system has involutive symbol, and hence the system is also involutive. Indeed in our case the Mansfield Prolongation Theorem [19] can be used to determine a bound for that order. Once the system is involutive, then (after a generic change of coordinates if necessary), an analytic existence and uniqueness theorem can be given.

If no generic change of coordinates is needed, then the analytic data, is specified by a finite number of analytic functions on a hyper-surface which is left invariant by one of the frame derivation operators.

One of the novelties of the commutative Riquier Theory, is that an analytic existence and uniqueness theorem is obtained in the infinite case without needing to change into generic coordinates. We sketch below some partial results obtained in the infinite non-commutative case.

We can generalize the Riquier analyticity theorem to the non-commuting infinite case by assuming that the $\widetilde{\partial}_i$ are lined up with an analytical system of coordinates. In particular we make the hypothesis **(H)** that there exist m analytical functions X_i and a neighborhood $N(x^0)$ of the expansion point x^0 satisfying:

- $\widetilde{\partial}_i X_j = 0$ in $N(x^0)$ if $i \neq j$
- The Jacobian of $(X_1, \ldots, X_m)$ does not vanish in $N(x^0)$

This is less stringent than assuming that the $\widetilde{\partial}_i$ are associated to a set of coordinates since in that case we would have $\widetilde{\partial}_i X_j = 1$, if $i = j$ and 0 otherwise.

Theorem 9.7 *Let $\prec$ be a Riquier ranking compatible with the total degree ordering. Suppose that the frame $\widetilde{\partial}_i$ satisfies **(H)**. Consider a non-commutative Riquier Basis $\mathcal{M}$ and analytic initial data specification ϕ. Then the unique formal power series solution thus defined is also analytic.*

The proof consists in transforming our problem into a commuting derivative problem in the set of coordinates X_i where we can apply the Riquier theorem.

A sketch of the straightforward proof follows.

Sketch of proof

- Without loss of generality set $x^0 = 0$.

- Introduce new commuting derivations $\widehat{\partial}_i$ which are based on the system of coordinates X.
- Prove that there exist m scalar functions $A_i(x)$ analytic at x^0 such that $\widetilde{\partial}_i = A_i\widehat{\partial}_i$ with $A_i(0) \neq 0$.
- Any derivation $\widetilde{\partial}^\alpha$ can be rewritten in terms of $A_1^{\alpha_1} \cdots A_m^{\alpha_m}\widehat{\partial}^\alpha$ plus a finite sum of terms $f_\beta\widehat{\partial}^\beta$ where f_β is an analytic function and β strictly "divides" α (i.e. $\beta_k \leq \alpha_k$ for $1 \leq k \leq m$).
- Replace the $\widetilde{\partial}_i$ by $\widehat{\partial}_i$ in $\mathcal{M}$ and obtain an orthonomic system $\widehat{\mathcal{M}}$. The set of leaders of $\widehat{\mathcal{M}}$ coincide with the ones in $\mathcal{M}$ by replacing the $\widetilde{\partial}_i$ with $\widehat{\partial}_i$.
- The analytic initial condition specification ϕ of $\mathcal{M}$ defines an analytic initial condition specification $\widehat{\phi}$ of $\widehat{\mathcal{M}}$.
- Since the problem has been reduced to the commutative case, $\widehat{\phi}$ has a geometrical meaning. In particular this demonstrates that the dependant variables and some of their derivatives have been fixed to analytic functions on unions of sub-manifolds of the form $X_i = 0$.
- Since the ranking is Riquier and compatible with total order, the commutative Riquier analyticity theorem applies.

□

It is interesting to note that the condition **(H)** is always satisfied in the case $m = 2$ (the case $m = 1$ is obvious).

In general the condition **(H)** of the previous subsection is not satisfied as the following example shows:

Example 9.8

$$\begin{cases} \widetilde{\partial}_1 = \partial_x + \partial_z \\ \widetilde{\partial}_2 = \partial_y + z\partial_z \\ \widetilde{\partial}_3 = \partial_z \end{cases}$$

Since $\widetilde{\partial}_1\widetilde{\partial}_2 - \widetilde{\partial}_2\widetilde{\partial}_1 = \widetilde{\partial}_3$, *the relations* $\widetilde{\partial}_1 X_3 = 0$ *and* $\widetilde{\partial}_2 X_3 = 0$ *imply* $\widetilde{\partial}_3 X_3 = 0$. *Thus,* X_3 *cannot generate a system of coordinates.*

Thus in the general case the above geometric interpretation is lost and it is not straightforward to adapt the proof of the analyticity theorem ([29], [13]) to the non-commuting case. Indeed, the use of majorizing functions to prove convergence of the formal series rely on commuting derivatives. The analyticity results obtained from prolonging the system to involution indicates there is a reasonable chance of proving a suitable non-commutative Riquier analyticity theorem in the non-commutative case.

Acknowledgements

GJR and FL thank Ian Lisle for helpful discussions. GJR thanks Colin Rust for early discussions regarding this work. GJR and YZ thank Mark Giesbrecht for early discussions. We thank Wenquin Zhou for work and discussions surrounding Theorem 9.6.

References

[1] M.A. Akivis and B.A. Rosenfeld, *Élie Cartan (1869-1951)*, Translations Math. Monographs, vol. 123, American Math. Soc., Providence, R.I., 1993.

[2] François Boulier, Daniel Lazard, François Ollivier, and Michel Petitot, *Representation for the radical of a finitely generated differential ideal*, proceedings of ISSAC'95 (Montréal, Canada) (A.H.M. Levelt, ed.), ACM Press, 1995, pp. 158–166.

[3] T. Becker, V. Weispfenning, and H. Kredel, *Gröbner Bases, a Computational Approach to Commutative Algebra*, corrected second printing ed., vol. 141 of Graduate Texts in Mathematics. Springer, New York, 1998.

[4] M. Boutin, *Numerically invariant signature curves*, Int. J. Computer Vision **40** (2000), 235–248.

[5] C.J. Budd and A. Iserles, *Geometric Integration: numerical integration of differential equations on manifolds*, Phil. Trans. Roy. Soc: London A **357** (1999), 945–956.

[6] É. Cartan, *La Méthode du Repère Mobile, la Théorie des Groupes Continus, et les Espaces Généralisés*, Exposés de Géométrie No. 5, Hermann, Paris 1935.

[7] M. Fels and P.J. Olver, *Moving Coframes. I. A practical algorithm*, Acta. Appl. Math. **51** (1998), 161–213.

[8] M. Fels and P.J. Olver, *Moving Coframes. II. Regularization and theoretical foundations*, Acta. Appl. Math. **55** (1999), 127–208.

[9] M. Giesbrecht, G. Reid and Y. Zhang, *Non-commutative Gröbner Bases in Poincaré-Birkhoff-Witt Extensions*, Proc. of the Fifth International Workshop on Computer Algebra in Scientific Computing (CASC 2002, Yalta, Ukraine), edited by V.G. Ganzha, E.W. Mayr & E.V. Vorozhtsov (Pub: Technical University of Munich) 97–106.

[10] P.A. Griffiths, *On Cartan's method of Lie Groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. **41** (1974), 775–814.

[11] E. Hairer, C. Lubich and G. Wanner, *Geometric Numerical Integration*, Springer-Verlag, New York, 2002.

[12] W. Hereman, *Review of symbolic software for Lie symmetry analysis*, Math. and Comp. Modelling **25** (1997) 115–132.

[13] M. Janet, *Sur les systèmes d'équations aux dérivées partielles*, J. de Math, 3(1920), 65-151.

[14] I.A. Kogan, *Inductive construction of moving frames*, Contemp. Math. **285** (2001), 157–170.

[15] E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York 1973.

[16] I.G. Lisle, *Equivalence Transformations for Classes of Differential equations*, PhD Thesis (University of British Columbia, 1992). Available at `www.ise.canberra.edu.au/MathStat/StaffPages/LisleDissertation.pdf`.

[17] I.G. Lisle and G.J. Reid (2000), *Symmetry classification using invariant moving frame*, ORCCA Technical Report TR-00-08. Available at `http://www.orcca.on.ca/TechReports/2000/TR-00-08.html`

[18] E.L. Mansfield, *Algorithms for symmetric differential systems*, Foundations of Computational Math.(2001) 1:335-383.

[19] E.L. Mansfield, *A simple criterion for involutivity*, J. London Math. Soc. (2) **54**, (1996) 2:323–345.

[20] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics **107**, Springer-Verlag, New York, 1993.

[21] P.J. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.

[22] P.J. Olver, *Geometric foundations of numerical algorithms and symmetry*, Appl. Alg. Engin. Comput. **11** (2001), 417–436.

[23] P.J. Olver, *Moving Frames*, pre-print, University of Minnesota (see http://www.math.umn.edu/˜olver/xtra.html).

[24] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[25] Jean-François Pommaret, *Systems of Partial Differential Equations and Lie pseudogroups*, Gordon and Breach science publishers Inc., 1978.

[26] G. Reid, *Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution*, Euro. J. Appl. Maths., 2(1991), 293-318.

[27] G.J. Reid, I.G. Lisle, A. Boulton, and A.D. Wittkopf, *Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs*, proceedings of ISSAC'92 (Berkeley, CA, USA) (Paul S. Wang, ed.), ACM Press, 1992, pp. 63–68.

[28] G. Reid, A. Wittkopf and A. Boulton, *Reduction of systems of nonlinear partial differential equations to simplified involutive forms*, Euro. J. Appl. Maths., **7** 1996, 604–635.

[29] Charles Riquier. *Les systèmes d'équations aux dérivées partielles*, Gauthier–Villars, Paris, 1910.

[30] C. J. Rust, *Rankings on derivatives for elimination algorithms and formal solvability of analytic partial differential equations*, Ph.D. thesis, University of Chicago, 1998.

[31] C.J. Rust, G.J. Reid and A.D. Wittkopf, *Existence and uniqueness theorems for formal power series solutions of analytic differential systems*, Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation, 105–112 (electronic), ACM, New York, 1999.

[32] M. Spivak, A Comprehensive Introduction to Differential Geometry, Publish or Perish, Houston, Texas, 2ed, 1979.

[33] A. Tresse, *Sur les invariants différentiels des groupes continus de transformations*, Acta. Math. **18**, (1894) 1–88.