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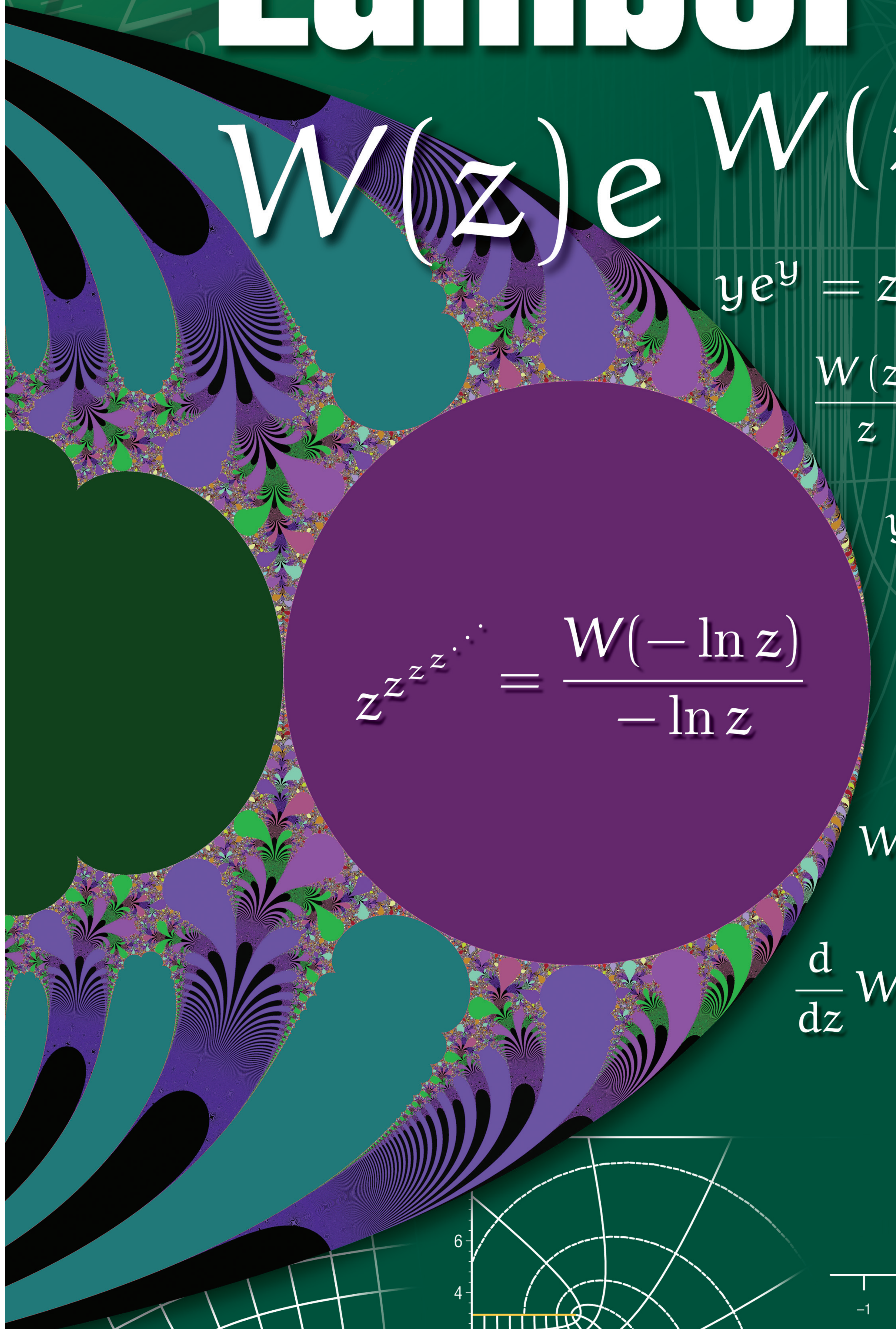
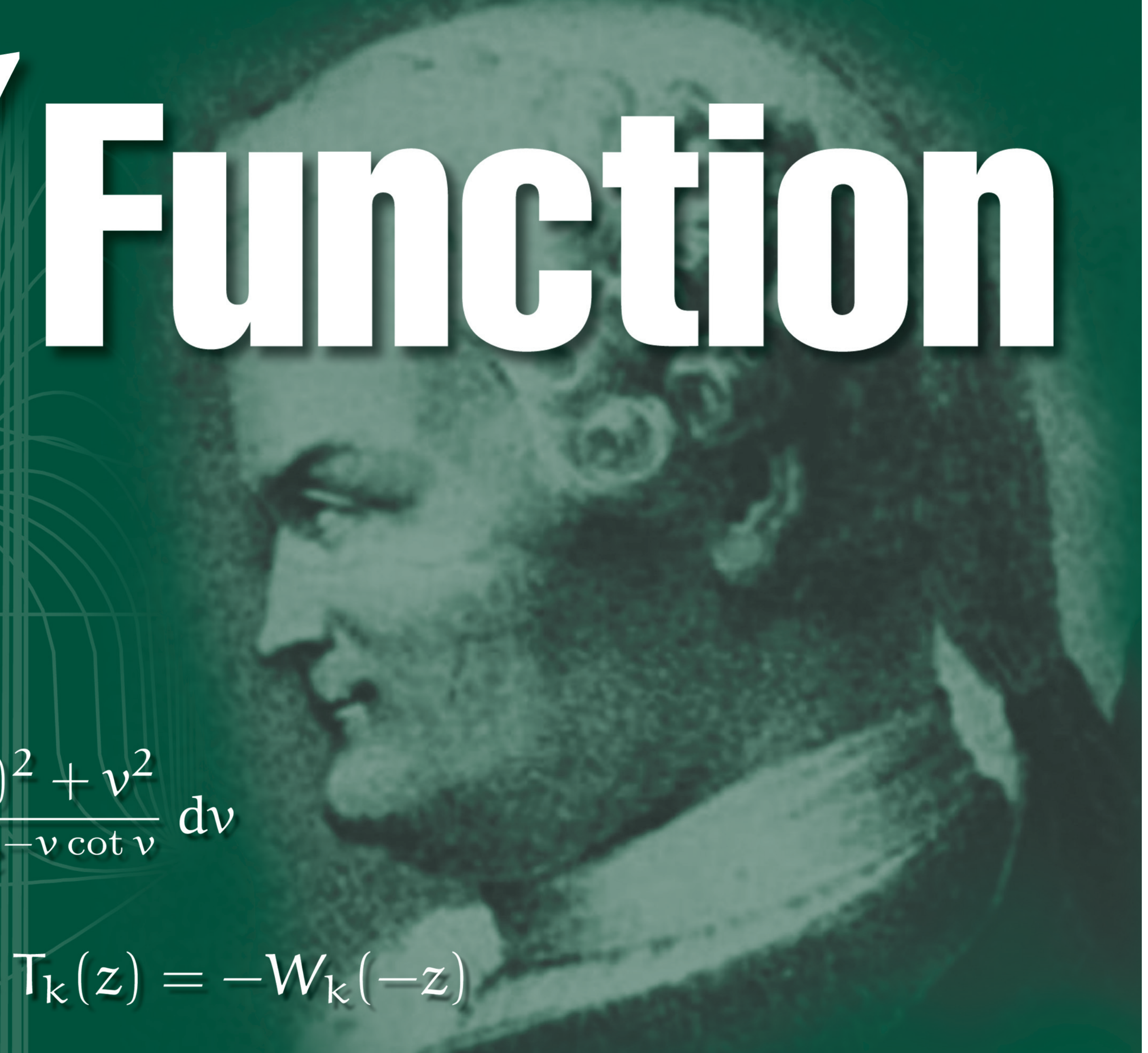
The Lambert W Function





$$W(z)e^{W(z)} = z$$

$$ye^{y} = z \iff y = W_k(z)$$

$$\frac{W(z)}{z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1 - v\cot v)^2 + v^2}{z + v\csc v\, e^{-v\cot v}}\, dv$$

$$ye^{-y} = z \iff y = T_k(z) = -W_k(-z)$$

$$z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}} = \frac{W(-\ln z)}{-\ln z}$$

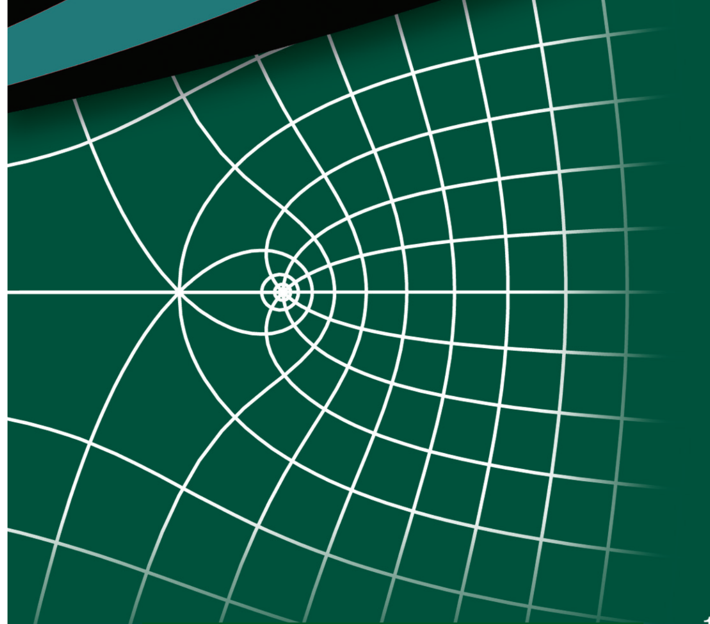
A Fractal Related to W

Each colour represents a cycle length in the iteration $a_{n+1} = z^{a_n}$, with $a_0 = 1$. A pixel at coordinate $\zeta = x + iy$ where $\zeta = T(\ln z)$ is given the colour corresponding to the length of the attracting cycle.

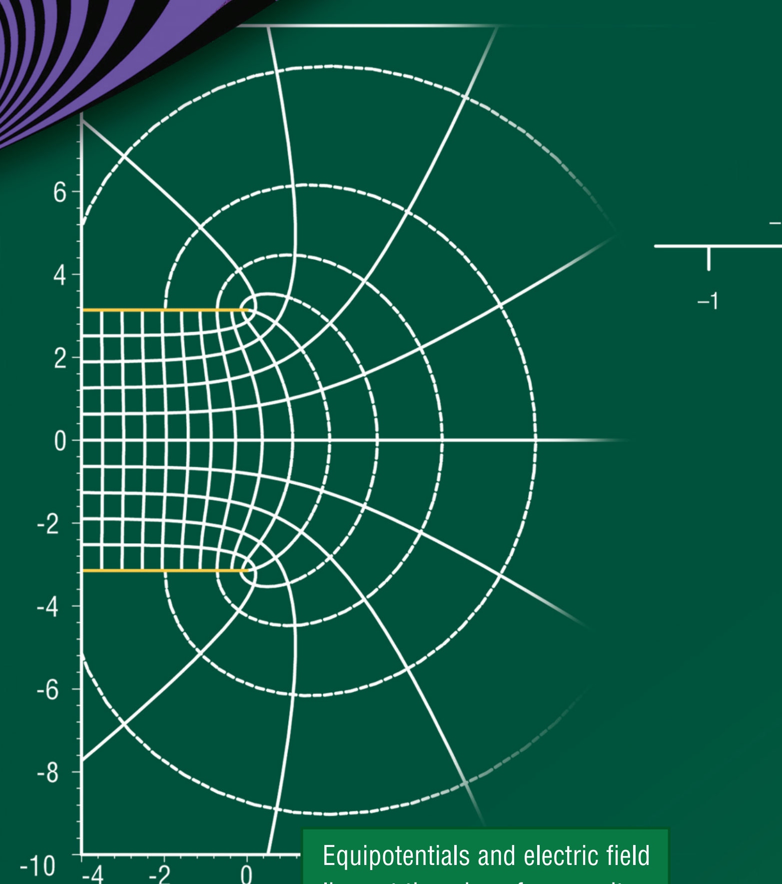
$$W(z) = \sum_{n\geq 1} \frac{(-n)^{n-1}}{n!} z^n$$

$$\frac{d}{dz}W(z) = \frac{W(z)}{z(1 + W(z))}$$

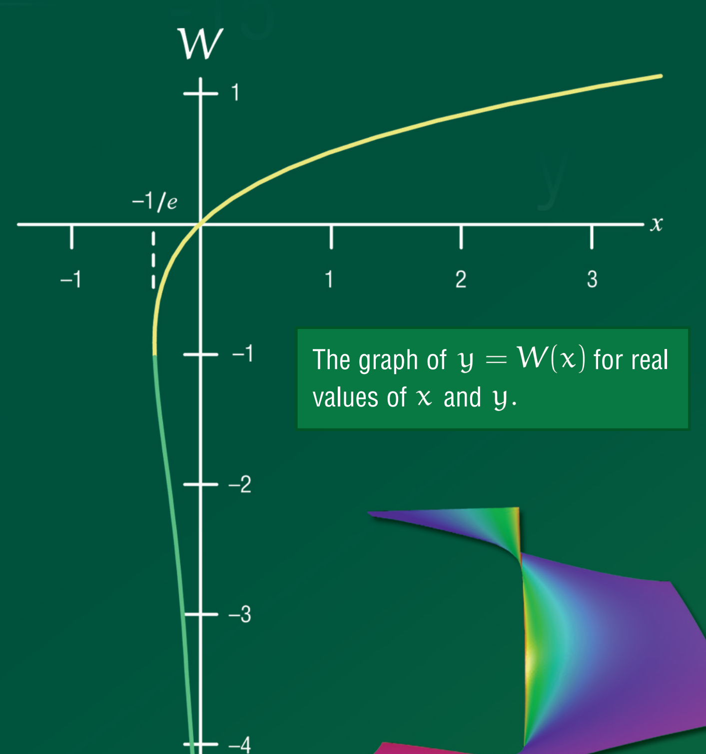
if $z \neq 0, -1/e$



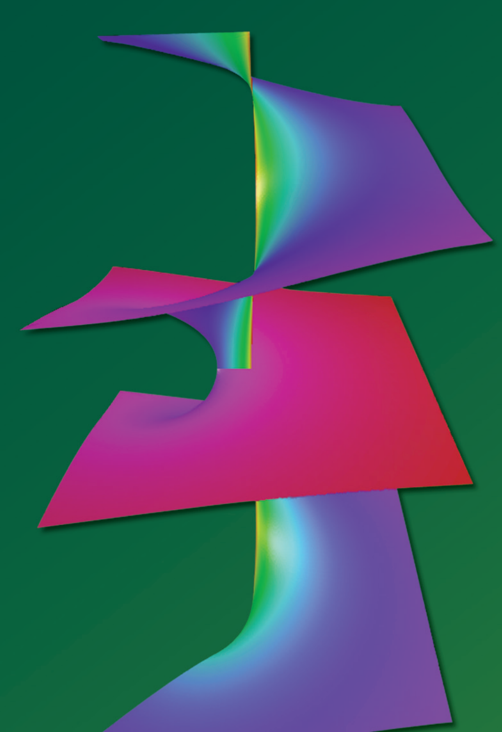
Images of circles and rays under the maps $z \mapsto W_k(z)$. Equivalently, images of horizontal and vertical lines under the map $z \mapsto \omega(z) = W_{K(z)}(e^z)$.



Equipotentials and electric field lines at the edge of a capacitor consisting of two charged thin plates a distance 2π apart.
$\zeta = z - 1 + \omega(z-1)$



The graph of $y = W(x)$ for real values of x and y.



A portion of the Riemann surface for $W(z)$, drawn by plotting a surface with height $\mathrm{Im}\,(W(x+iy))]$ at coordinates (x, y) and colouring the surface with Re $(W(x+iy))$; the apparent intersection on the line $-1/e \leq x \leq 0,\ y = 0$ is of surfaces with different colours and therefore not a true intersection.

$$\int W(z)\, dz = \frac{z\left(W^2(z) - W(z) + 1\right)}{W(z)} + C$$

$$\int_0^{\infty} x^{s-1} W(x)\, dx = \frac{(-s)^{-s}\,\Gamma(s)}{s} \quad \text{if } -1 < \mathrm{Re}(s) < 0$$

$$\int 2 \sin W(x)\, dx = \left(x + \frac{x}{W(x)}\right) \sin W(x) - x \cos W(x) + C$$

$$\int_0^{\infty} e^{-st} W(e^t)\, dt = s^{s-2}\Gamma(1-s, sW(1)) + \frac{W(1)}{s} \quad \text{if } \mathrm{Re}(s) > 0$$

Johann Heinrich Lambert

Johann Heinrich Lambert was born in Mulhouse on the 26th of August, 1728, and died in Berlin on the 25th of September, 1777. His scientific interests were remarkably broad. The self-educated son of a tailor, he produced fundamentally important work in number theory, geometry, statistics, astronomy, meteorology, hygrometry, pyrometry, optics, cosmology and philosophy. Lambert was the first to prove the irrationality of π. He worked on the parallel postulate, and also introduced the modern notation for the hyperbolic functions.

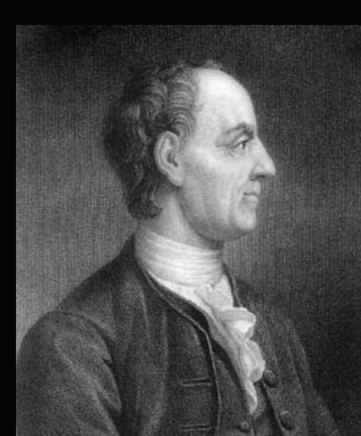
In a paper entitled "Observationes Variae in Mathesin Puram", published in 1758 in *Acta Helvetica*, he gave a series solution of the trinomial equation, $x^m + px = q$, for x. His method was a precursor of the more general Lagrange inversion theorem. This solution intrigued his contemporary, Euler, and led to the discovery of the Lambert W function.

Lambert wrote Euler a cordial letter on the 18th of October, 1771, expressing his hope that Euler would regain his sight after an operation; he explains in this letter how his trinomial method extends to series reversion.

The Lambert W function is *implicitly elementary.* That is, it is implicitly defined by an equation containing only elementary functions. The Lambert W function is not, itself, an elementary function. It is also not a *Liouvillian* function, which means that it is not expressible as a finite sequence of exponentiations, root extractions, or antidifferentiations (quadratures) of any elementary function.

The Lambert W function has been applied to solve problems in the analysis of algorithms, the spread of disease, quantum physics, ideal diodes and transistors, black holes, the kinetics of pigment regeneration in the human eye, dynamical systems containing delays, and in many other areas.

Leonhard Euler



Leonhard Euler was born on the 15th of April, 1707, in Basel, Switzerland, and died on the 18th of September, 1783, in St. Petersburg, Russia. Half his papers were written in the last fourteen years of his life, even though he had gone blind.

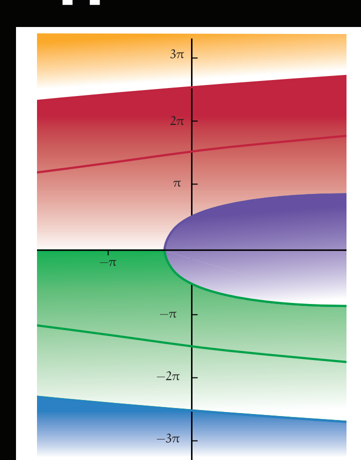
Euler was the greatest mathematician of the 18th century, and one of the greatest of all time. His work on the calculus of variations has been called "the most beautiful book ever written", and Pierre Simon de Laplace exhorted his students: "Lisez Euler, c'est notre maître â tous", advice that is still profitable today.

Many functions and concepts are named after him, including the Euler totient function, Eulerian numbers, the Euler-Lagrange equations, and the "eulerian" formulation of fluid mechanics. The mathematical formulae on this poster are typeset in the Euler font, designed by Hermann Zapf to evoke the flavour of excellent human handwriting.

Lambert's series solution of his trinomial equation, which Euler rewrote as $x^{\alpha} - x^{\beta} = (\alpha - \beta)vx^{\alpha+\beta}$, led to the series solution of the transcendental equation $x \ln x = v$. This was the earliest known occurrence of the series for the function now called the Lambert W function.

$$x^y = y^x \iff y = -\frac{x}{\ln x} W_k\left(-\frac{\ln x}{x}\right)$$

Hippias of Elis



Hippias of Elis lived, travelled and worked around 460 BC, and is mentioned by Plato. The Quadratrix (or trisectrix) of Hippias is the first curve ever named after its inventor. As drawn in the picture here, its equation is $x = -y\cot y$. This curve can be used to square the circle and to trisect the angle. Since these classical problems are unsolvable by straightedge and compass, we therefore conclude that the construction of the Quadratrix is impossible under that restriction. The Quadratrix is also the image of the real axis under the map $z \mapsto W_k(z)$, and the parts of the curve corresponding to the negative real axis delimit the ranges of the branches of W. We have here coloured the ranges of the different branches of W with different colours.

Sir Edward Maitland Wright $\omega(z) = W_{K(z)}(e^z)$

Sir Edward Maitland Wright was born the 1st of January, 1906. He is the co-author with G. H. Hardy of the classic book *An Introduction to the Theory of Numbers.* His main contributions to the study of the Lambert W function were a systematic way of computing its complex values, a series expansion of a related function about its branch points, the application of W to enumeration problems, and the application of W to the study of the stability of the solutions of linear and nonlinear delay differential equations. He was Professor of Mathematics, then Principal and Vice-Chancellor, of Aberdeen University (1936-1976).

$$z = \ln e^z + 2\pi i K(z)$$

R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth, "On the Lambert W Function", Advances in Computational Mathematics, volume 5, 1996, pp. 329-359

www.orcca.on.ca/LambertW

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Western